Solutions 6

1. a) We have

$$A_{n+1} - A_n = \mathbb{E}(X_{n+1} \,|\, \mathcal{F}_n) - X_n = \mathbb{E}(X_n + \xi_{n+1} \,|\, \mathcal{F}_n) - X_n = X_n + \mathbb{E}(\xi_{n+1}) - X_n = 2p - 1,$$

that is, $A_n = n(2p-1)$ and $M_n = X_n - A_n = X_n - n(2p-1)$.

b) Here, we have

$$\begin{aligned} A_{n+1} - A_n &= \mathbb{E}(X_{n+1} \,|\, \mathcal{F}_n) - X_n = \mathbb{E}(S_{n+1}^2 \,|\, \mathcal{F}_n) - S_n^2 = \mathbb{E}(S_n^2 + 2S_n\,\xi_{n+1} + \xi_{n+1}^2 \,|\, \mathcal{F}_n) - S_n^2 \\ &= S_n^2 + 2S_n \mathbb{E}(\xi_{n+1} \,|\mathcal{F}_n) + \mathbb{E}(\xi_{n+1}^2 \,|\, \mathcal{F}_n) - S_n^2 = 2S_n\,\mathbb{E}(\xi_{n+1}) + \mathbb{E}(\xi_{n+1}^2) = 1, \end{aligned}$$

so $A_n = n$ and $M_n = X_n - A_n = S_n^2 - n$. Notice that we have already proven in Homework 5, Exercise 3, that $(M_n = S_n^2 - n,\, n \in \mathbb{N})$ is a martingale.

2. a) T is neither a stopping tiome, nor it is bounded.

b) T is an unbounded stopping time.

c) T is not a stopping time, but it is bounded.

d) T is a bounded stopping time.

3. a) We need to check that for all $n \in \mathbb{N}$, $\{T = n\} \in \mathcal{F}_n$. Indeed,

$$\{T = n\} = \{|S_i| < a,\ \forall 1 \le i \le n-1 \quad \text{and} \quad |S_n| \ge a\} = \cap_{i=1}^{n-1}\{|S_i| < a\} \cap \{|S_n| \ge a\} \in \mathcal{F}_n,$$

since each $\{|S_i| < a\} \in \mathcal{F}_i \subset \mathcal{F}_n$.

b) Applying Doob's optional stopping theorem, we have $\mathbb{E}(M_T) = \mathbb{E}(M_0) = 0$, so $\mathbb{E}(S_T^2 - T) = 0$. This implies that $\mathbb{E}(T) = \mathbb{E}(S_T^2) = a^2$, since at time T, $|S_T| = a$ (by definition of what T is).

4. Let $m \in \mathbb{N}$ and U be an $\mathcal{F}_m$-measurable and bounded random variable. Let us also define

$$H_n = \begin{cases} U, & \text{if } n = m+1, \\ 0, & \text{otherwise.} \end{cases}$$

Then $(H_n,\, n \in \mathbb{N})$ is predictable and for $m < N$, we have by assumption taht M_m is $\mathcal{F}_m$-measurable and also that

$$0 = \mathbb{E}((H \cdot M)_N) = \mathbb{E}(U(M_{m+1} - M_m)).$$

Therefore, $M_m = \mathbb{E}(M_{m+1}|\mathcal{F}_m)$, so $(M_n,\, n \in \mathbb{N})$ is a martingale.