1. a) We have
\[ A_{n+1} - A_n = \mathbb{E}(X_{n+1} \mid \mathcal{F}_n) - X_n = \mathbb{E}(X_n + \xi_{n+1} \mid \mathcal{F}_n) - X_n = X_n + \mathbb{E}(\xi_{n+1}) - X_n = 2p - 1, \]
that is, \( A_n = n(2p - 1) \) and \( M_n = X_n - A_n = X_n - n(2p - 1). \)

b) Here, we have
\[ A_{n+1} - A_n = \mathbb{E}(X_{n+1} \mid \mathcal{F}_n) - X_n = \mathbb{E}(S_{2n+1} \mid \mathcal{F}_n) - S_n^2 = \mathbb{E}(S_{n+1}^2 + 2S_n \xi_{n+1} + \xi_{n+1}^2 \mid \mathcal{F}_n) - S_n^2 \]
\[ = S_n^2 + 2S_n \mathbb{E}(\xi_{n+1} \mid \mathcal{F}_n) + \mathbb{E}(\xi_{n+1}^2 \mid \mathcal{F}_n) - S_n^2 = 2S_n \mathbb{E}(\xi_{n+1}) + \mathbb{E}(\xi_{n+1}^2) = 1, \]
so \( A_n = n \) and \( M_n = X_n - A_n = S_n^2 - n. \) Notice that we have already proven in Homework 5, Exercise 3, that \((M_n = S_n^2 - n, n \in \mathbb{N})\) is a martingale.

2. a) \( T \) is neither a stopping time, nor it is bounded.

b) \( T \) is an unbounded stopping time.

c) \( T \) is not a stopping time, but it is bounded.

d) \( T \) is a bounded stopping time.

3. a) We need to check that for all \( n \in \mathbb{N}, \{ T = n \} \in \mathcal{F}_n. \) Indeed,
\[ \{ T = n \} = \{|S_i| < a, \forall 1 \leq i \leq n - 1 \} \text{ and } |S_n| \geq a = \cap_{i=1}^{n-1} \{|S_i| < a\} \cap \{|S_n| \geq a\} \in \mathcal{F}_n, \]
since each \( \{|S_i| < a\} \in \mathcal{F}_i \subset \mathcal{F}_n. \)

b) Applying Doob’s optional stopping theorem, we have \( \mathbb{E}(M_T) = \mathbb{E}(M_0) = 0, \) so \( \mathbb{E}(S_T^2 - T) = 0. \) This implies that \( \mathbb{E}(T) = \mathbb{E}(S_T^2) = a^2, \) since at time \( T, |S_T| = a \) (by definition of what \( T \) is).

4. Let \( m \in \mathbb{N} \) and \( U \) be an \( \mathcal{F}_m \)-measurable and bounded random variable. Let us also define
\[ H_n = \begin{cases} U, & \text{if } n = m + 1, \\ 0, & \text{otherwise}. \end{cases} \]
Then \((H_n, n \in \mathbb{N})\) is predictable and for \( m < N, \) we have by assumption that \( M_m \) is \( \mathcal{F}_m \)-measurable and also that
\[ 0 = \mathbb{E}((H \cdot M)_N) = \mathbb{E}(U(M_{m+1} - M_m)). \]
Therefore, \( M_m = \mathbb{E}(M_{m+1} \mid \mathcal{F}_m), \) so \((M_n, n \in \mathbb{N})\) is a martingale.