Solutions 5

1. a) comes from the fact that $\mathbb{E}(\mathbb{E}(X|\mathcal{F})) = \mathbb{E}(X)$.

b) comes from the fact that M_n is $\mathcal{F}_n$-measurable.

c) Let us show this by induction. We know that $\mathbb{E}(M_{n+1}|\mathcal{F}_n) = M_n$. Suppose now that $\mathbb{E}(M_{n+m}|\mathcal{F}_n) = M_n$ for some $m \geq 1$. Since $\mathcal{F}_n \subset \mathcal{F}_{n+m}$, we have

$$\mathbb{E}(M_{n+m+1}|\mathcal{F}_n) = \mathbb{E}(\mathbb{E}(M_{n+m+1}|\mathcal{F}_{n+m})|\mathcal{F}_n) = \mathbb{E}(M_{n+m}|\mathcal{F}_n) = M_n,$$

by the martingale property and the induction hypothesis. The conclusion therefore holds.

2. a) By Jensen's inequality, we know that

$$\mathbb{E}(\varphi(X_{n+1})|\mathcal{F}_n) \geq \varphi(\mathbb{E}(X_{n+1}|\mathcal{F}_n)).$$

We also know that $\mathbb{E}(X_{n+1}|\mathcal{F}_n) \geq X_n$, since (X_n) is a submartingale. The function φ needs therefore to be also increasing in order to ensure that

$$\mathbb{E}(\varphi(X_{n+1})|\mathcal{F}_n) \geq \varphi(X_n).$$

b) In particular, (X_n^2) need not necessarily be a submartingale, whereas $(\exp(X_n))$ is.

3. a) Clearly, since $S_0 = 0, S_1 = \xi_1, \ldots, S_n = \xi_1 + \ldots + \xi_n$, $(S_0, S_1, \ldots, S_n)$ is a function of $(\xi_1, \ldots, \xi_n)$, so the natural filtration

$$\mathcal{F}_n^S = \sigma(S_0, \ldots, S_n) \subset \sigma(\xi_1, \ldots, \xi_n),$$

On the other hand, $\xi_n = S_n - S_{n-1}$ for all $n \geq 1$, so $(\xi_1, \ldots, \xi_n)$ is also a function of $(S_0, S_1, \ldots, S_n)$ and therefore,

$$\sigma(\xi_1, \ldots, \xi_n) \subset \sigma(S_0, \ldots, S_n) = \mathcal{F}_n^S.$$

b) For all n, $S_n^2 - n$ is clearly integrable and also $\mathcal{F}_n$-measurable. There remains therefore to prove that

$$\mathbb{E}(S_{n+1}^2 - (n+1) \,|\, \mathcal{F}_n) = S_n^2 - n, \quad \forall n \in \mathbb{N}.$$

i.e.

$$\mathbb{E}(S_{n+1}^2 \,|\, \mathcal{F}_n) = S_n^2 + 1, \quad \forall n \in \mathbb{N}.$$

This holds true, since

$$\begin{aligned}\mathbb{E}(S_{n+1}^2 \,|\, \mathcal{F}_n) &= \mathbb{E}((S_n + \xi_{n+1})^2 \,|\, \mathcal{F}_n) = \mathbb{E}(S_n^2 + 2S_n\xi_{n+1} + \xi_{n+1}^2 \,|\, \mathcal{F}_n) \\ &= S_n^2 + 2S_n\mathbb{E}(\xi_{n+1} \,|\, \mathcal{F}_m) + \mathbb{E}(\xi_{n+1}^2 \,|\, \mathcal{F}_n) = S_n^2 + 2S_n\mathbb{E}(\xi_{n+1}) + \mathbb{E}(\xi_{n+1}^2) = S_n^2 + 1.\end{aligned}$$

4. a) Yes. Let us denote by $(\xi_n,\, n \geq 1)$ the process defined as $\xi_n = +1$ if the n^{th} coin toss falls on "heads" and $\xi_n = -1$ if the n^{th} coin toss falls on "tails". The bet H_n at time n is given by

$$H_1 = 1, \quad H_{n+1} = \begin{cases} 0, & \text{if } \xi_n = +1, \\ 2H_n, & \text{if } \xi_n = -1, \end{cases}$$

and the gain of the player at time n is given by

$$G_n = \sum_{i=1}^{n} H_i\, \xi_i.$$

This process can indeed be shown to be a martingale. Let $\mathcal{F}_0 = \{\emptyset, \Omega\}$ and $\mathcal{F}_n = \sigma(\xi_1, \ldots, \xi_n)$. Then following the same procedure as in exercise 3.a), it can be shown that $(\mathcal{F}_n, n \in \mathbb{N})$ is the natural filtration of the process $(G_n,\, n \in \mathbb{N})$. So G_n is $\mathcal{F}_n$-measurable, $\mathbb{E}(|G_n|) < \infty$ for all n (this requires a small check) and

$$\begin{aligned} \mathbb{E}(G_{n+1}|\mathcal{F}_n) &= \mathbb{E}(G_n + H_{n+1}\, \xi_{n+1}|\mathcal{F}_n) = \mathbb{E}(G_n|\mathcal{F}_n) + \mathbb{E}(H_{n+1}\, \xi_{n+1}|\mathcal{F}_n) \\ &\overset{(*)}{=} G_n + H_{n+1}\, \mathbb{E}(\xi_{n+1}|\mathcal{F}_n) = G_n + H_{n+1}\, \mathbb{E}(\xi_{n+1}) = G_n + 0 = G_n, \end{aligned}$$

where $(*)$ comes from the fact that both G_n and H_{n+1} are $\mathcal{F}_n$-measurable.

b) Suppose that the first "heads" comes out at time $T = n \geq 1$. Then the gain of the player is

$$-1 - 2 - 4 - \ldots - 2^{n-1} + 2^n = 1,$$

which does not depend on n. So whatever the time T, the gain of the player at this time is 1 franc.

c) There is a seeming contradiction, as $G_0 = 0$ and $G_T = 1$, so $\mathbb{E}(G_T) = 1 \neq 0 = \mathbb{E}(G_0)$. But as we will see, stopping a martingale at a random unbounded time is not innocent...