Solutions 10

1. We have $\lim_{n\to\infty} \sum_{i=1}^{2^n} \left(B\left(\frac{it}{2^n}\right) - B\left(\frac{(i-1)t}{2^n}\right)\right)^4$

$$\leq \left(\lim_{n\to\infty} \sup_{1\leq i\leq 2^n} \left(B\left(\frac{it}{2^n}\right) - B\left(\frac{(i-1)t}{2^n}\right)\right)^2 \right) \cdot \left(\lim_{n\to\infty} \sum_{i=1}^{2^n} \left(B\left(\frac{it}{2^n}\right) - B\left(\frac{(i-1)t}{2^n}\right)\right)^2 \right).$$

The first term of the above product converges to zero a.s., since (B_t) is continuous and therefore uniformly continuous on $[0,t]$. The second term converges to the quadratic variation of (B_t) in $[0,t]$, which is a.s. finite, since it is equal to t a.s. Therefore, the product converges to zero a.s.

2. We show that $(M_t^2 - \mathbb{E}(M_t^2))$ is a martingale. Indeed, let $t > s \geq 0$:

$$\begin{aligned} \mathbb{E}(M_t^2 - M_s^2|\mathcal{F}_s) &= \mathbb{E}((M_t - M_s)^2 + 2\,(M_t - M_s)\,M_s|\mathcal{F}_s) = \mathbb{E}((M_t - M_s)^2) \\ &= \mathbb{E}(M_t^2) - \mathbb{E}(M_s^2) - 2\,\mathbb{E}((M_t - M_s)\,M_s) = \mathbb{E}(M_t^2) - \mathbb{E}(M_s^2). \end{aligned}$$

3. a) We know that $M_t - M_s \geq 0$ a.s., for all $t > s \geq 0$, and we also know that

$$\mathbb{E}(M_t - M_s) = \mathbb{E}(\mathbb{E}(M_t - M_s|\mathcal{F}_s)) = \mathbb{E}(\mathbb{E}(M_t|\mathcal{F}_s) - M_s) = \mathbb{E}(M_s - M_s) = 0,$$

so $M_t = M_s$ a.s. for all $t > s \geq 0$, i.e. $M_t = M_0$ a.s. for all $t \in \mathbb{R}_+$.

b) Let us compute, for $t > s$,

$$\begin{aligned} \mathbb{E}((M_t - M_s)^2) &= \mathbb{E}(M_t^2 - 2M_tM_s + M_s^2) = \mathbb{E}(\mathbb{E}(M_t^2 - 2M_tM_s + M_s^2|\mathcal{F}_s)) \\ &= \mathbb{E}(\mathbb{E}(M_t^2|\mathcal{F}_s) - 2\mathbb{E}(M_t|\mathcal{F}_s)M_s + M_s^2) = \mathbb{E}(M_s^2 - 2M_s^2 + M_s^2) = 0. \end{aligned}$$

where we have used the assumption that $\mathbb{E}(M_t^2|\mathcal{F}_s) = M_s^2$. Therefore, $M_t = M_s = M_0$ a.s. for all $t > s \geq 0$.