Quiz

1. Let X, Y be two random variables and

$$\mathcal{F} = \sigma(X, Y) \qquad \mathcal{G} = \sigma(X + Y) \qquad \mathcal{H} = \sigma(X, X + Y)$$

For each assertion a) to d) below, say whether it is true or not.

a) $\mathcal{F} \subset \mathcal{G}$ b) $\mathcal{G} \subset \mathcal{F}$

c) $\mathcal{F} \subset \mathcal{H}$ d) $\mathcal{H} \subset \mathcal{F}$

e) Does any of the above answers change if we moreover assume that X and Y are independent?

2. Let X and Y be two independent random variables.

a) Compute $\mathbb{E}((X+Y)^2|Y)$ (there should not remain any conditional expectation in the answer).

Let us now moreover assume that X, Y are i.i.d. and such that $\mathbb{P}(X = \pm 1) = \mathbb{P}(Y = \pm 1) = \frac{1}{2}$.

b) Compute $\mathbb{E}((X+Y)^2|Y)$ in this particular case.

c) Compute also $\mathbb{E}(X|X+Y)$ in this particular case.

3. Let $(\xi_n,\, n \in \mathbb{N})$ be a sequence of i.i.d.$\sim \mathbb{N}(0,1)$ random variables. Let $M_0 = \xi_0$ and $M_n = \xi_n + \xi_{n-1}$ for $n \geq 1$. Let also $(\mathcal{F}_n,\, n \in \mathbb{N})$ be the natural filtration of $(M_n,\, n \in \mathbb{N})$.

a) Is $(M_n,\, n \in \mathbb{N})$ a martingale with respect to $(\mathcal{F}_n,\, n \in \mathbb{N})$?

b) Is $(M_n,\, n \in \mathbb{N})$ a Markov process with respect to $(\mathcal{F}_n,\, n \in \mathbb{N})$?

c) Is $(M_n,\, n \in \mathbb{N})$ a Gaussian process?

d) Compute the mean and the covariance of $(M_n,\, n \in \mathbb{N})$.

4. Let $(S_n,\, n \in \mathbb{N})$ be the simple symmetric random walk on $\mathbb{Z}$ and $(\mathcal{F}_n,\, n \in \mathbb{N})$ be its natural filtration.

a) Is the process $(S_n^4,\, n \in \mathbb{N})$ a submartingale with respect to $(\mathcal{F}_n,\, n \in \mathbb{N})$?

b) Is the process $(S_n^4 - n,\, n \in \mathbb{N})$ a submartingale with respect to $(\mathcal{F}_n,\, n \in \mathbb{N})$?

Hint: Recall that $(x+y)^4 = x^4 + 4x^3y + 6x^2y^2 + 4xy^3 + y^4$.

c) Show that $\mathbb{E}(S_{n+1}^4) = \mathbb{E}(S_n^4) + 6n + 1$ and deduce the value of $\mathbb{E}(S_n^4)$ by induction on n.

Hint: Recall also that $\sum_{i=1}^{n-1} i = \frac{n(n-1)}{2}$.

d) Compute $\lim_{n\to\infty} \dfrac{\mathbb{E}(S_n^4)}{n^2}$. Can you make a parallel with something you already know?

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5. Let $(X_t,\, t \in \mathbb{R}_+)$ be a continuous-time process and $(\mathcal{F}_t,\, t \in \mathbb{R}_+)$ be its natural filtration. Let also $t > s \geq 0$.

a) Does $\mathbb{E}(X_t|\mathcal{F}_s) \geq X_s$ a.s. imply that $X_t \geq X_s$ a.s.?

b) Does $\mathbb{E}(X_t|\mathcal{F}_s) \geq X_s$ a.s. imply that $\mathbb{E}(X_t) \geq \mathbb{E}(X_s)$?

c) Does $\mathbb{E}(X_t|\mathcal{F}_s) = 0$ a.s. imply that $X_t = 0$ a.s.?

d) Does $\mathbb{E}(X_s|\mathcal{F}_s) = 0$ a.s. imply that $X_s = 0$ a.s.?

e) Does the property "$\mathbb{E}(X_t - X_s|\mathcal{F}_s) = 0$ a.s. for all $t > s \geq 0$" imply that the process X has independent increments?

6. Let $(B_t,\, t \in \mathbb{R}_+)$ be a standard Brownian motion.

a) What is the distribution of $2B_t - B_s$, for $t > s \geq 0$?

b) Is it true that $\dfrac{B_t^2}{t}$ has the same distribution as B_1^2, for all $t > 0$?

Among the following processes, which are submartingales?

c) $(\exp(B_t),\, t \in \mathbb{R}_+)$

d) $(\exp(-B_t),\, t \in \mathbb{R}_+)$

e) $(\exp(|B_t|),\, t \in \mathbb{R}_+)$

f) $(\exp(-|B_t|),\, t \in \mathbb{R}_+)$

7. Let $(M_t,\, t \in \mathbb{R}_+)$ be a continuous square-integrable martingale and $(\mathcal{F}_t,\, t \in \mathbb{R}_+)$ be its natural filtration. Let us moreover assume that $M_0 = 0$ a.s. and that $\langle M \rangle_t = t^2$ a.s., for all $t \in \mathbb{R}_+$. For each assertion a) to d) below, say whether it is true or not.

a) $\mathbb{E}(M_t^2) = t^2$, for all $t \geq 0$.

b) $M_t = t$ a.s., for all $t \geq 0$.

c) The trajectories of $(M_t,\, t \in \mathbb{R}_+)$ have bounded variation, a.s.

d) $\mathbb{E}(M_t^2 - M_s^2|\mathcal{F}_s) = t^2 - s^2$, for all $t > s \geq 0$.

e) Finally, compute the variance of $M_t - M_s$, for $t > s \geq 0$.

8. Let $(B_t,\, t \in \mathbb{R}_+)$ be a standard Brownian motion and $(X_t,\, t \in \mathbb{R}_+)$ be the process defined as $X_t = \exp(aB_t + bt)$ for $t \geq 0$, where $a, b \in \mathbb{R}$ are fixed.

a) Use Ito-Doeblin's formula to decompose the process $(X_t,\, t \in \mathbb{R}_+)$ into the sum of a martingale and a process with bounded variation.

b) Under which condition on the coefficients a and b is the process $(X_t,\, t \in \mathbb{R}_+)$ a submartingale?

c) Under which condition on the coefficients a and b is the process $(X_t,\, t \in \mathbb{R}_+)$ a martingale?

d) In the case where $(X_t,\, t \in \mathbb{R}_+)$ is a martingale, compute its quadratic variation.