Homework 7

Exercise 1. Let $(\xi_n,\, n \geq 1)$ be a sequence of i.i.d. centered random variables and let $(\mathcal{F}_n,\, n \geq 1)$ be the filtration defined as $\mathcal{F}_n = \sigma(\xi_1, \ldots, \xi_n)$, $n \geq 1$. Among the following processes $(X_n,\, n \geq 1)$, which are Markov processes with repspect to $(\mathcal{F}_n,\, n \geq 1)$? which are martingales with respect to $(\mathcal{F}_n,\, n \geq 1)$? (no formal justification needed; the answer suffices)

a) $X_n = \xi_n$, $n \geq 1$.

b) $X_1 = \xi_1$, $X_{n+1} = a\, X_n + \xi_{n+1}$, $n \geq 1$ ($a > 0$ fixed).

c) $X_1 = \xi_1$, $X_{n+1} = \xi_n + \xi_{n+1}$, $n \geq 1$.

d) $X_n = \max(\xi_1, \ldots, \xi_n)$, $n \geq 1$.

e) $X_1 = \xi_1$, $X_n = \sum_{i=1}^n (\xi_1 + \ldots + \xi_{i-1})\, \xi_i$, $n \geq 1$.

Exercise 2. An urban legend says: "two uncorrelated Gaussian random variables are necessarily independent". The exercise below shows that this is wrong.

Let X, Y be two centered Gaussian random variables, with variance 1.

a) Show that if X and Y are independent, then (X, Y) is a Gaussian vector, so $X + Y$ is Gaussian.

b) Let $\mathcal{E}$ be a random variable independent of X and such that $\mathbb{P}(\mathcal{E} = +1) = \mathbb{P}(\mathcal{E} = -1) = \frac{1}{2}$. Show that $Z = \mathcal{E}X$ is Gaussian, but that $X + Z$ is not, and therefore that (X, Z) is not a Gaussian vector.

c) Show also that X and Z are not independent, even though $\text{Cov}(X, Z) = 0$.

Exercise 3. Let $a \in \mathbb{R}$ and $X = (X_1, X_2, X_3)$ be a centered random vector (not necessarily Gaussian) such that

$$\mathbb{E}(X_1^2) = \mathbb{E}(X_2^2) = \mathbb{E}(X_3^2) = 1 \quad \text{and} \quad \mathbb{E}(X_1\, X_2) = \mathbb{E}(X_2\, X_3) = \mathbb{E}(X_1\, X_3) = a.$$

What values are allowed for the parameter a? In particular, can it be that

$$\mathbb{E}(X_1\, X_2) = \mathbb{E}(X_2\, X_3) = \mathbb{E}(X_1\, X_3) = -1?$$

Hint: Compute the eigenvalues of the covariance matrix of X.

Exercise 4. a) Let us assume that the vector X defined in Exercise 3 is Gaussian. For what values of a is the vector X degenerate? Describe the vector X in these cases.

b) Let $b \in [-1, +1]$ and $Y = (Y_1, Y_2, Y_3)$ be a centered Gaussian random vector whose covariance matrix is given by

$$K = \begin{pmatrix} 1 & b & b^2 \\ b & 1 & b \\ b^2 & b & 1 \end{pmatrix}$$

For what values of b is the vector Y degenerate? Describe the vector Y in these cases.