Homework 12

Exercise 1. Let $(B_t,\, t \in \mathbb{R}_+)$ be a standard Brownian motion with respect to $(\mathcal{F}_t,\, t \in \mathbb{R}_+)$. Let also

$$M_t = B_t^2 - t \quad \text{and} \quad N_t = \exp\left(B_t - \frac{t}{2}\right).$$

By Ex. 1 in Hw. 9, we already know that (M_t) and (N_t) are martingales with respect to $(\mathcal{F}_t)$. Write explicitly Doob's decomposition of the submartingales (M_t^2) and (N_t^2), and deduce from that the values of $\langle M \rangle_t$ and $\langle N \rangle_t$.

Hint: Write M_t^2 and N_t^2 as functions of t and B_t, and use Ito-Doeblin's formula.

Exercise 2. (Ornstein-Uhlenbeck's process)
Let $(B_t,\, t \in \mathbb{R}_+)$ be a standard Brownian motion and $a \in \mathbb{R}$ be fixed. Let also

$$X_t = \int_0^t e^{-a(t-s)}\, dB_s, \quad t \in \mathbb{R}_+.$$

a) By Ex. 2 in Hw. 11, we know that (X_t) is a Gaussian process. Compute its mean and covariance.

b) Is (X_t) a process with independent increments? A martingale? Justify your answer.

c) Show that (X_t) satisfies the following stochastic differential equation:

$$X_t = -a \int_0^t X_s\, ds + B_t \quad \text{and} \quad X_0 = 0.$$

Hint: Write X_t as the product $V_t M_t$, where $V_t = e^{-at} = \int_0^t (-ae^{-as})\, ds$ and $M_t = \int_0^t e^{as}\, dB_s$ and use the generalized Ito-Doeblin formula.

Exercise 3. a) Let $(B_t,\, t \in \mathbb{R}_+)$ be a standard Brownian motion. Let also $b < 0$, $c > 0$ and $T = \inf\{t \in \mathbb{R}_+ : B_t \notin]b, c[\}$ be the first time the process B exits from the interval $]b, c[$ (fact: T is a stopping time). Given that (B_t) is continuous, B_T can only possibly take two values, b or c, with probabilities p_b and p_c respectively. Compute p_b (numerical example: $c = 1$ and $b = -2$).

Hint: Use the optional stopping theorem (valid here even if T is not a bounded stopping time).

b) Redo the same exercise, replacing the standard Brownian motion (B_t) by the Ornstein-Uhlenbeck process (X_t) defined in Ex. 2.

Hint: Show first, using again the generalized Ito formula, that if $f : \mathbb{R} \to \mathbb{R}$ satisfies the following ordinary differential equation:

$$-ax\, f'(x) + \frac{1}{2} f''(x) = 0, \quad f(0) = 0, \quad f'(0) = 1.$$

then the process $(f(X_t))$ is a (continuous) martingale.

Numerical example ($c = 1$ and $b = -2$): give an explicit formula for the solution f of the above equation and determine for which values of $a \in \mathbb{R}^*$ is the probability p_b higher than in the Brownian case.