Homework 1

Exercise 1. Let $\Omega = \{1, \ldots, 6\}$ et $\mathcal{A} = \{\{1, 3, 5\}, \{1, 2, 3\}\}$.

a) Describe $\mathcal{F} = \sigma(\mathcal{A})$, the σ-field generated by $\mathcal{A}$.

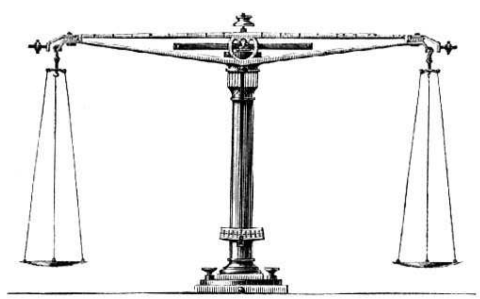
Hint: For a finite set Ω, the number of elements of a σ-field on Ω is always a power of 2.

b) Give the list of non-empty elements G of $\mathcal{F}$ such that

$$\text{if } F \in \mathcal{F} \text{ and } F \subset G, \text{ then } F = \emptyset \text{ or } G.$$

These elements are called the *atoms* of the σ-field $\mathcal{F}$. They form a *partition* of the set Ω and they also generate the σ-field $\mathcal{F}$.

Exercise 2. Using a traditional balance, three people (say A, B and C) try to measure the weight of an object, which we assume not to exceed 100g.



For this measure, A has weights of 20g and 50g; B has weights of 20g only and C has weights of 10g only. On the other hand, the number of weights available for each of them is unlimited.

Determine the amount of information that each person has on the weight of the object, and order theses informations. In particular, determine who is able to decide whether the weight of the object is between 40g and 50g or not.

Remark: One assumes that when measuring, all weights are on the same side of the balance, with the object on the other side.

Exercise 3. Let $\mathcal{F}$ be a σ-field defined on a set Ω. Using only the axioms given in the definition of a σ-field, show the following properties:

a) $\Omega \in \mathcal{F}$.

b) $\cap_{n=1}^{\infty} A_n \in \mathcal{F}$, if $(A_n)_{n=1}^{\infty} \subset \mathcal{F}$.

c) $B \backslash A \in \mathcal{F}$, if $A, B \in \mathcal{F}$.

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Exercise 4. Let $\Omega = [-1, 1[$ and $(X_i, i = 1, \ldots, 4)$ be a family of random variables on Ω defined as

$$X_i(\omega) = \begin{cases} 1 & \text{if } \frac{i-1}{4} \leq \omega < \frac{i}{4}, \\ (-1)^i & \text{if } -\frac{i}{4} \leq \omega < -\frac{i-1}{4}, \\ 0 & \text{otherwise.} \end{cases}$$

Describe the σ-field $\mathcal{F} = \sigma(X_i, i = 1, \ldots, 4)$ using its atoms (see exercise 1).