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School of Computer and Communication Sciences

Handout 8 Signal Processing for Communications
Homework 3 Solution March 15, 2010

PROBLEM 1. Evaluate the following integral,

$$\int_{-\pi}^{\pi} \sin(x)\delta(-2x-\pi)\,dx.$$

$$\sin(x)\delta(-2x-\pi) = \sin(-\frac{\pi}{2})\delta(-2x-\pi) = -\delta(-2x-\pi).$$

Now, assume that $y = -2x - \pi$. So $dy = -2dx$, and y would be in the range of $[\pi, -3\pi]$ since x is in the range of $[-\pi, \pi]$. So, we will have:

$$-\int_{-\pi}^{\pi} \delta(-2x-\pi)dx = -\int_{\pi}^{-3\pi} \delta(y)(-\frac{1}{2}dy) = \frac{1}{2}(-\int_{-3\pi}^{\pi} \delta(y)dy) = -\frac{1}{2}.$$

Notice that $\int_a^b(.) = -\int_b^a(.)$.

You can also solve this problem in this way: In the last part, assume that $z = -y$, so it would be:

$$-\int_{\pi}^{-3\pi} \delta(y)(-\frac{1}{2}dy) = \frac{1}{2}\int_{-\pi}^{3\pi} \delta(-z)(-dz) = -\frac{1}{2}\int_{-\pi}^{3\pi} \delta(z)dz = -\frac{1}{2}.$$

Remember that since δ is an even function, we will have $\delta(z) = \delta(-z)$ which is used in the last part.

PROBLEM 2. Do the following vectors form a basis in $\mathcal{R}^4$?

$(3.14, 2, 1, -2)$, $(1.73, 2, -0.5, -2.7)$, $(0.3, 0.3, -0.5, -0.73)$, $(1, 0.3, -0.5, -0.73)$, $(1, 1, 1, 1)$

Since we are in $\mathcal{R}^4$, the basis should consists of 4 linear independent vectors. So we can easily conclude that these 5 vectors can not be considered as a basis. In fact, at least one of them surely can be written as a linear combination of the others.

PROBLEM 3. Consider a length-N signal $x[n]$, $n = 0, \ldots, N-1$; what is the length-N signal $y[n]$ obtained as

$$y[n] = \text{DFT}\{\text{DFT}\{x[n]\}\}?$$

Remember that since we are in DFT, domain, all m, n, k in the following are from 0 to $(N-1)$. Remember also that $W_N^{kn} = e^{-j\frac{2\pi}{N}kn}$.

$$\begin{aligned}
y_1[k] &= \text{DFT}\{x[n]\} = \sum_{n=0}^{N-1} x[n] W_N^{kn}. \\
y[m] &= \text{DFT}\{y_1[k]\} = \sum_{k=0}^{N-1} y_1[k] W_N^{mk} \\
&= \sum_{k=0}^{N-1} (\sum_{n=0}^{N-1} x[n] W_N^{kn}) W_N^{mk} \\
&= \sum_{k=0}^{N-1} \sum_{n=0}^{N-1} x[n] W_N^{k(n+m)} \\
&= \sum_{n=0}^{N-1} x[n] \sum_{k=0}^{N-1} W_N^{k(n+m)} \\
&= \sum_{n=0}^{N-1} x[n] N \delta((n+m) \, mod \, N) \\
&= N x[-m \, mod \, N].
\end{aligned}$$

In fact $y[0] = Nx[0], y[1] = Nx[N-1], y[2] = Nx[N-2], ..., y[N-1] = Nx[1]$.

PROBLEM 4. Derive the formula for the DFT of the length-N signal $x[n] = \cos(\frac{2\pi}{N} Ln + \phi)$.

$y[k] = \text{DFT}\{x[n]\}$ and $k = 0, ..., (N-1)$.

$$\begin{aligned}
y[k] &= \sum_{n=0}^{N-1} \cos(\frac{2\pi}{N} Ln + \phi) W_N^{nk} \\
&= \frac{1}{2} \sum_{n=0}^{N-1} [\exp(j(\frac{2\pi}{N} Ln + \phi)) + \exp(-j(\frac{2\pi}{N} Ln + \phi))] W_N^{nk} \\
&= \frac{1}{2} e^{j\phi} \sum_{n=0}^{N-1} W_N^{n(k-L)} + \frac{1}{2} e^{-j\phi} \sum_{n=0}^{N-1} W_N^{n(k+L)} \\
&= \frac{N}{2} e^{j\phi} \delta((k-L) \, mod \, N) + \frac{N}{2} e^{-j\phi} \delta((k+L) \, mod \, N).
\end{aligned}$$

PROBLEM 5. Compute the DFT of the length-4 signal $x[n] = \{a, b, c, d\}$. For which values of a, b, c, d is the DFT real?

$$y[k] = \text{DFT}\{x[n]\} = \sum_{n=0}^{3} W_N^{kn} = a + b \, W_N^k + c \, W_N^2 k + d \, W_N^3 k$$

Since $N = 4$, we can write $W_N^{3k} = W_N^{-k}$ because $W_N^{k+TN} = W_N^k$ for all integers T. W_N^{2k} is also equal to $(-1)^k$ because $W_4^2 = e^{-j\pi} = -1$. If $b = d$ then we can simplify the above equation in this way:

$$
\begin{aligned}
y[k] = \mathrm{DFT}\{x[n]\} &= \sum_{n=0}^{3} W_N^{kn} = a + b\,W_N^{k} + c\,(-1)^k + d\,W_N^{-k} \\
&= a + c\,(-1)^k + 2\,b\,\mathrm{Re}(W_N^{k}).
\end{aligned}
$$

In this case $y[k]$ would be real if all a, b, c, d are real and $b = d$. Notice that it is not the whole answer, because there are also some other complex values for a, b, c, d that cause the real DFT; but, there is no such a nice condition as above.

Problem 6. Consider the FT pair of $x[n]$ and $X(e^{j\omega})$. Find $Y(e^{j\omega})$ in terms $X(e^{j\omega})$ for the following signals:

(a) $y[n] = nx[n]$

$$
\begin{aligned}
\frac{dX(e^{j\omega})}{d\omega} &= \frac{d}{d\omega}\sum_n x[n]e^{-j\omega n} \\
&= \sum_n x[n]\frac{d}{d\omega}e^{-j\omega n} \\
&= \sum_n -jn\,x[n]e^{-j\omega n} \\
&= -j\sum_n nx[n]e^{-j\omega n} \\
&= -jY(e^{j\omega})
\end{aligned}
$$

$$
\rightarrow Y(e^{j\omega}) = j\frac{dX(e^{j\omega})}{d\omega}
$$

(b) $y[n] = x[n - n_d]$, n_d is an integer.

$$
\begin{aligned}
Y(e^{j\omega}) &= \sum_n x[n - n_d]e^{-j\omega n} \\
&= \sum_m x[m]e^{-j\omega(m+n_d)} \\
&= e^{-j\omega n_d}\sum_m x[m]e^{-j\omega m} \\
&= e^{-j\omega n_d}X(e^{j\omega})
\end{aligned}
$$

$$
\rightarrow Y(e^{j\omega}) = e^{-j\omega n_d}X(e^{j\omega}).
$$

(c) $y[n] = -x[n]$.

$$Y(e^{j\omega}) = \sum_n -x[n]e^{-j\omega n} = -\sum_n x[n]e^{-j\omega n} = -X(e^{j\omega})$$
$$\rightarrow Y(e^{j\omega}) = -X(e^{j\omega}).$$

(d) $y[n] = -3x[4n-7]$.

In this item, you should be more careful. I try to split the system to some subsystems. Assume we have a subsystem $y_1[n] = x[n-7]$. It just shifts the input by 7 to the right. At second step, consider $y_2[n] = y_1[4n]$. It is actually a down-sampling subsystem. It means that it chooses one sample from each 4-tuple samples of y_1. At the last step, $y_3[n] = -3y_2[n]$ which is such as an amplifier (with negative sign!). Before solving this problem, let's try to find the FT of the output of a down-sampling system by 2, i.e. $y[n] = x[2n]$. (I uses Z-transform for more convenience)

$$\begin{aligned} Y(z) &= \sum_n y[n]z^{-n} \\ &= \sum_n x[2n]z^{-n} \\ &= \sum_{n=2m} x[n]z^{-\frac{n}{2}} \\ &= \sum_n x[n]z^{-\frac{n}{2}}(\frac{1+(-1)^n}{2}) \\ &= \frac{1}{2}\sum_n x[n](z^{\frac{1}{2}})^{-n} + \frac{1}{2}\sum_n x[n](-z^{\frac{1}{2}})^{-n} \\ &= \frac{1}{2}[X(z^{\frac{1}{2}}) + X(-z^{\frac{1}{2}})]. \end{aligned}$$

So we can write

$$\begin{aligned} Y(e^{j\omega}) &= \frac{1}{2}[X(e^{j\frac{\omega}{2}}) + X(-e^{j\frac{\omega}{2}})] \\ &= \frac{1}{2}[X(e^{j\frac{\omega}{2}}) + X(e^{j\frac{\omega-2\pi}{2}})] \end{aligned}$$

.

For down sampling by 4 it is sufficient to use the previous system twice. It is obvious that if $\tilde{y}[n] = x[4n]$, it is equivalent to $\tilde{y}[n] = y[2n]$. If you compute in similar way you will obtain this formula for down-sampling by 4.

$$\tilde{Y}(e^{j\omega}) = \frac{1}{4}\,[X(e^{j\frac{\omega}{4}}) + X(e^{j\frac{\omega-2\pi}{4}}) + X(e^{j\frac{\omega-4\pi}{4}}) + X(e^{j\frac{\omega-6\pi}{4}})].$$

Now, let us back to our own problem.

$$\begin{aligned} Y_3(e^{j\omega}) &= -3Y_2(e^{j\omega}) \\ &= -3\,[\,\frac{1}{4}\,[Y_1(e^{j\frac{\omega}{4}}) + Y_1(e^{j\frac{\omega-2\pi}{4}}) + Y_1(e^{j\frac{\omega-4\pi}{4}}) + Y_1(e^{j\frac{\omega-6\pi}{4}})]\,] \end{aligned}$$

where $Y_1(e^{j\omega}) = e^{-j\,7\,\omega}X(e^{j\omega})$.

(e) $y[n] = nx[-n+1]$.

$$
\begin{aligned}
Y(e^{j\omega}) &= \sum_n nx[-n+1]e^{-j\omega n} \\
&= \sum_m (1-m)x[m]e^{-j\omega(1-m)} \\
&= e^{-j\omega}\sum_m (1-m)x[m]e^{j\omega m} \\
&= e^{-j\omega}[\sum_m x[m]e^{-j(-\omega)m} - \sum_m mx[m]e^{-j(-\omega)m}] \\
&= e^{-j\omega}[X(e^{-j\omega}) - j\frac{dX(e^{j\omega})}{d\omega}|_{\omega=-\omega}] \\
&= Y(e^{j\omega}) = e^{-j\omega}[X(e^{-j\omega}) + j\frac{dX(e^{-j\omega})}{d\omega}]
\end{aligned}
$$

I should mention that I used the result of item (a) directly.

PROBLEM 7. Consider the discrete signal $x[n]$ with support N, take the DTFT of $x[n]$ to obtain $X(e^{j\omega})$. Instead of computing the inverse DTFT of $X(e^{j\omega})$ to obtain $x[n]$, we first sample the $X(e^{j\omega})$ in $[0, 2\pi]$ at N points and we call it $\tilde{Y}[n]$ and then take the inverse of N-point DFT of the $\tilde{Y}[n]$ to obtain $y[n]$. Relate $y[n]$ to $x[n]$.

$$
\begin{aligned}
X(e^{j\omega}) &= \sum_{n=-\infty}^{+\infty} x[n]e^{-j\omega n} \\
&= \sum_{n=0}^{N-1} x[n]e^{-j\omega n}
\end{aligned}
$$

$$
\begin{aligned}
\tilde{Y}[k] &= X(e^{j\omega})|_{\omega=\frac{2\pi}{N}k,(k=0,..,N-1)} \\
&= \sum_{n=0}^{N-1} x[n]e^{-j\frac{2\pi}{N}kn}.
\end{aligned}
$$

$$
\begin{aligned}
y[m] &= \frac{1}{N}\sum_{k=0}^{N-1}\tilde{Y}[k]W_N^{-km} \\
&= \frac{1}{N}\sum_{k=0}^{N-1}(\sum_{n=0}^{N-1}x[n]e^{-j\frac{2\pi}{N}kn})W_N^{-km} \\
&= \frac{1}{N}\sum_{n=0}^{N-1}x[n]\sum_{m=0}^{N-1}W_N^{k(n-m)} \\
&= \frac{1}{N}\sum_{n=0}^{N-1}x[n]N\delta((n-m)\,mod\,N) \\
&= \sum_{n=0}^{N-1}x[n]\delta((n-m)modN) \\
&= x[m\,mod\,N].
\end{aligned}
$$

Since m is from 0 to $(N-1)$, we can conclude $y[m] = x[m]$.

We haven't covered the LTI systems in the class, but you have seen it in previous courses. In the following problems, we will review LTI systems.

PROBLEM 8. For each of following systems determine whether the system is (1) stable, (2) causal, (3) linear, (4) time invariant and (5) memoryless.

Let's look at these properties more precisely. Assume $y[n]$ is the output of system $T(x[n])$.

A system is stable if $|x[n]| < \infty \rightarrow |y[n]| < \infty$.

It is causal if the output at time n, $y[n]$, depends on only the input samples at previous or present times i.e. depends on $x[k]$'s for $k \leq n$.

Linearity means $T[ax_1[n] + bx_2[n]] = aT[x_1[n]] + bT[x_2[n]] = ay_1[n] + by_2[n]$.

Time invariant means when you shift the input, the output also will be shifted. In fact if $x_2[n] = x_1[n - n_d]$ then $y_2[n] = y_1[n - n_d]$.

Memorylessness means that $T[x[n]]$ just depends on the input at the present time n, i.e. $x[n]$. So, we can conclude that whenever a system is memoryless, it would also be causal but the inverse is not true because the output of a causal system at time n might be dependent also on input samples at previous times, so there is no guaranty that such a system is also memoryless.

(a) $T(x[n]) = x[n]g[n]$, with $g[n]$ given.

It is *stable on condition that* $g[n]$ *is bounded.* It is *causal* and *memoryless* because $y[n]$ depends on $x[n]$ not other $x[k], k \neq n$. It is also *linear* because $T[ax_1[n] + bx_2[n]] =$

$[ax_1[n] + bx_2[n]]g[n] = ax_1[n]g[n] + bx_2[n]g[n] = aT[x_1[n]] + bT[x_2[n]]$. It is *not time invariant* in general case because $g[n]$ can be variant in time. In fact, if $x_2[n] = x_1[n-n_d]$ then $y_2[n] = x_2[n]g[n] = x_1[n-n_d]g[n]$ while $y_1[n-n_d] = x_1[n-n_d]g[n-n_d]$ and clearly $y_2[n] \neq y_1[n-n_d]$.

(b) $T(x[n]) = x[Mn]$, with M a positive integer.

It is *stable* and *linear* trivially, because $|x[Mn]| < \infty \rightarrow |y[n]| < \infty$ and $T[ax_1[n] + bx_2[n]] = ax_1[Mn] + bx_2[Mn] = aT[x_1[n]] + bT[x_2[n]]$. it is *not causal* because $y[n]$ depends on future signal samples (M is positive). So, It is *not memoryless* also (why?). Let's check if it is time invariant or not. If $x_2[n] = x_1[n-n_d]$, then, $y_2[n] = x_2[Mn] = x_1[Mn-n_d]$ while $y_1[n-n_d] = x_1[M(n-n_d)] = x_1[Mn - Mn_d]$ which is not equal to $y_1[n-n_d]$. So, it is *not time invariant.*

(c) $T(x[n]) = \sum_{k=n_0}^{n} x[k]$.

It is *not stable* because the output is the summation of input samples from n_0 up to n and it can go to infinity even if the input is bounded. It is *linear* because summation is a linear operator. it is *causal* because $y[n]$ just depends on $x[n_0], x[n_0+1], ..., x[n]$ and not future samples such as $x[n+1], ...$ It is *not memoryless* because as you can see $y[n]$ depends on some other input samples which are not for the present time. It is *not time invariant*, because if for example $x_1[n] = u[-n+n_0-1]$ where $u[n]$ is a step function, the output will be $y_1[n] = 0$ for $n > n_0$, while if we shift the input by 1 to the right, the output will be $y_2[n] = 1$ for $n > n_0$ which is not shifted version of $y_1[n]$.

(d) $T(x[n]) = \sum_{k=n-n_0}^{n+n_0} x[k]$.

It is *linear* because the summation is a linear operator. It is also *stable*, because the summation is taken over a limited interval and the output can not go to the infinity unless the input is not bounded. It is *not causal* because the output $y[n]$ depends on $x[k]$ for some $k > n$. So it is *not memoryless.* It is *not time invariant* because if $x_2[n] = x_1[n-n_d]$, then $y_2[n] = \sum_{k=n-n_0}^{n+n_0} x_1[n-n_d] = \sum_{k=n-n_d-n_0}^{n-n_d+n_0} x_1[n]$, while $y_1[n-n_d] = \sum_{k=n-n_d-n_0}^{n-n_d+n_0} x_1[n-n_d]$, and $y_1[n-n_d] \neq y_2[n]$. Notice: it is assumed that n_0 is a positive integer.

(e) $T(x[n]) = e^{x[n]}$.

It is *not linear*, because exponential function is appeared. It is *stable*, because $y[n]$ goes to infinity just when $x[n]$ goes to infinity. In fact, $e^{x[n]}$ can not be equal to infinity unless $x[n]$ is infinity. It is *memoryless* and so *causal.* It is *time invariant* which means the behavior of the system does not change through the time. you can easily check that $y_2[n] = y_1[n-n_d]$ because $e^{x_2[n]} = e^{x_1[n-n_d]}$ when $x_2[n] = x_1[n-n_d]$.

(f) $T(x[n]) = x[n] + 3u[n+1]$.

It is *not linear* because $T[x_1[n] + x_2[n]] = x_1[n] + x_2[n] + 3u[n+1]$, while $T[x_1[n]] + T[x_2[n]] = x_1[n] + x_2[n] + 6u[n+1]$. It is *stable*, *memoryless* and *causal*; Be careful that $y[n]$ does not depend on $x[k]$ for $k \neq n$. although there is $u[n+1]$ as second term in formula, but it is not related to the input. It is *not time invariant* because $y_2[n] = x_2[n]+3u[n+1] = x_1[n-n_d]+3u[n+1]$ and $y_1[n-n_d] = x_1[n-n_d]+3u[n-n_d+1]$ and $y_2[n] \neq y_1[n-n_d]$.

(g) $T(x[n]) = x[-n]$.

It is *linear* because $T[x_1[n] + x_2[n]] = x_1[-n] + x_2[-n] = T[x_1[n]] + T[x_2[n]]$. It is *stable* but it is *not causal* and *not memoryless* (just consider the case that you want to compute the output at negative times!). It is also *not time invariant* because $y_2[n] = x_2[-n] = x_1[-n - n_d]$, while $y_1[n - n_d] = x_1[-(n - n_d)] = x_1[-n + n_d]$ and so $y_2[n] \neq y_1[n - n_d]$.