ÉCOLE POLYTECHNIQUE FÉDÉRALE DE LAUSANNE
School of Computer and Communication Sciences

Handout 4 Signal Processing for Communications
Homework 2 Solution March 8, 2010

PROBLEM 1. 1. Show that the set of all ordered n-tuples $[a_1, a_2, \ldots, a_n]$ with the natural definition for the sum:

$$[a_1, a_2, \ldots, a_n] + [b_1, b_2, \ldots, b_n] = [a_1 + b_1, a_2 + b_2, \ldots, a_n + b_n]$$

and the multiplication by a scalar:

$$\alpha[a_1, a_2, \ldots, a_n] = [\alpha a_1, \alpha a_2, \ldots, \alpha a_n]$$

form a vector space. Give its dimension and find a basis.

We have to check the axioms for vector spaces. Addition must be associative

$$[a_1, a_2, \ldots, a_n] + ([b_1, b_2, \ldots, b_n] + [c_1, c_2, \ldots, c_n]) =$$

$$[a_1 + (b_1 + c_1), a_2 + (b_2 + c_2), \ldots, a_n + (b_n + c_n)] =$$
$$[(a_1 + b_1) + c_1, (a_2 + b_2) + c_2, \ldots, (a_n + b_n) + c_n] =$$
$$([a_1, a_2, \ldots, a_n] + [b_1, b_2, \ldots, b_n]) + [c_1, c_2, \ldots, c_n]$$

Addition must be commutative

$$[a_1, a_2, \ldots, a_n] + [b_1, b_2, \ldots, b_n] = [a_1 + b_1, a_2 + b_2, \ldots, a_n + b_n] =$$

$$[b_1, b_2, \ldots, b_n] + [a_1, a_2, \ldots, a_n]$$

There is a zero vector $\vec{0}$ which satisfies $\vec{v} + \vec{0} = \vec{v}$ for all $\vec{v}$

$$[a_1, a_2, \ldots, a_n] + [0, 0, \ldots, 0] = [a_1 + 0, a_2 + 0, \ldots, a_n + 0] = [a_1, a_2, \ldots, a_n]$$

Additive inverses must be in the vector space

$$[a_1, a_2, \ldots, a_n] + [-a_1, -a_2, \ldots, -a_n] = \vec{0}$$

Finally, we must show that the vector space must be closed under addition and multiplication by a scalar

$$\alpha \cdot [a_1, a_2, \ldots, a_n] + \beta \cdot [b_1, b_2, \ldots, b_n] =$$

$$[\alpha a_1, \alpha a_2, \ldots, \alpha a_n] + [\beta b_1, \beta b_2, \ldots, \beta b_n] =$$
$$[\alpha a_1 + \beta b_1, \alpha a_2 + \beta b_2, \ldots, \alpha a_n + \beta b_n]$$

which clearly is in the vector space

This proves that the set of all ordered n-tuples forms a vector space.

2. Show that the set of signals of the form $y(x) = a\cos(x) + b\sin(x)$ (for arbitrary a, b), with the usual addition and multiplication by a scalar, form a vector space. Give its dimension and find a basis.

As for the above question, you can show that the axioms for vector spaces hold for our kind of signals.

Associativity:

$$y_1(x) + (y_2(x) + y_3(x)) =$$
$$a_1\cos(x) + b_1\sin(x) + (a_2\cos(x) + b_2\sin(x) + a_3\cos(x) + b_3\sin(x)) =$$
$$(a_1 + a_2 + a_3)\cos(x) + (b_1 + b_2 + b_3)\sin(x) =$$
$$(a_1\cos(x) + b_1\sin(x) + a_2\cos(x) + b_2\sin(x)) + a_3\cos(x) + b_3\sin(x) =$$
$$(y_1(x) + y_2(x)) + y_3(x)$$

Commutativity:

$$y_1(x) + y_2(x) = a_1\cos(x) + b_1\sin(x) + a_2\cos(x) + b_2\sin(x) =$$

$$y_2(x) + y_1(x)$$

The function $0(x) = 0\cos(x) + 0\sin(x)$ satisfies the condition for the zero element

Inverses :

$$-y(x) = (-a)\cos(x) + (-b)\sin(x)$$

Closure:

$$\alpha \cdot y_1(x) + \beta \cdot y_2(x) =$$
$$\alpha a_1\cos(x) + \alpha b_1\sin(x) + \beta a_2\cos(x) + \beta b_2\sin(x) =$$
$$(\alpha a_1 + \beta a_2)\cos(x) + (\alpha b_1 + \beta b_2)\sin(x)$$

We will show that $\mathrm{B} = \{\cos(x), \sin(x)\}$ forms a basis. It must satisfy the following requirements :

- they span the whole space $a\cos(x) + b\sin(x)$;
- they are be independent, that is, you cannot write a function based on the other;
- they must be minimal, if you remove one of them, the remaining ones do not span the whole space.

To show that $\cos(x)$ and $\sin(x)$ are independent, you can proceed as follows : we want that

$$a \cdot \cos(x) + b \cdot \sin(x) = 0, \forall x \in \mathbb{R}$$

implies that $a = 0$ and $b = 0$.

3. Are the four diagonals of a cube orthogonal?

There are four diagonals in a cube. If they were pairwise orthogonal, then they would form a basis of $\mathbb{R}^3$. But we know that the dimension of $\mathbb{R}^3$ is three, and all bases have three vectors. Therefore, the diagonals cannot be orthogonal.

4. Express the discrete-time impulse $\delta[n]$ in terms of the discrete-time unit step $u[n]$ and conversely.

We can write down the relationship between the two signals as follows :

$$\sum_{k=-\infty}^{n} \delta[k] = u[n]$$

and

$$u[n] - u[n-1] = \delta[n]$$

Those two functions behave in a very similar manner to their continuous counterparts, where the sum is replaced by the integral, and the difference is replaced by the derivative.

5. Show that any function $f(t)$ can be written as the sum of an odd and an even function, i.e. $f(t) = f_o(t) + f_e(t)$ where $f_o(-t) = -f_o(t)$ and $f_e(-t) = f_e(t)$.

We can write the following equations :

$$\begin{cases} f(t) = f_o(t) + f_e(t) \\ f(-t) = -f_o(t) + f_e(t) \end{cases}$$

Summing the two equations, and dividing by 2, we get :

$$f_e(t) = \frac{f(t) + f(-t)}{2}$$

And the difference, followed by division by 2 gives:

$$f_o(t) = \frac{f(t) - f(-t)}{2}$$

Problem 2. Let $\{x(k)\}$, $k = 0, \ldots, N-1$, be a basis for a space S. Prove that any vector $z \in S$ is uniquely represented in this basis.

Hint. Prove by contradiction.

The fact that any vector $\vec{z}$ can be represented in the basis $\left\{\vec{x}^{(k)}\right\}_{\{k=0,\ldots,N-1\}}$ follows by the definition of a basis. We need to prove that the representation is unique: Suppose that $\vec{z}$ has two distinct representations $\{\alpha_k\}_{\{k=0,\ldots,N-1\}} \neq \{\beta_k\}_{\{k=0,\ldots,N-1\}}$. That is,

$$\vec{z} = \sum_{k=0}^{N-1} \alpha_k \vec{x}^{(k)}, \vec{z} = \sum_{k=0}^{N-1} \beta_k \vec{x}^{(k)}$$

We can thus write

$$\vec{0} = \vec{z} - \vec{z} = \vec{z} = \sum_{k=0}^{N-1} (\alpha_k - \beta_k)\vec{x}^{(k)} \neq \vec{0}$$

a contradiction. Therefore, $\vec{z}$ is uniquely represented in the basis.

PROBLEM 3. Assume v and w are two vectors in the vector space. Prove the triangular inequality for each v and w.

$$||v + w|| \leq ||v|| + ||w||.$$

Hint. Expand $||v + w||^2$ and use Cauchy-Schwarz inequality.

$$||v + w||^2 = \langle v + w, v + w\rangle$$
$$= |v|^2 + \langle v, w\rangle + \langle w, v\rangle + |w|^2$$
$$\leq |v|^2 + 2|\langle v, w\rangle| + |w|^2$$
$$\leq |v|^2 + 2|v||w| + |w|^2$$

(by the Cauchy-Schwarz Inequality)

$$= (\|v\| + \|w\|)^2$$

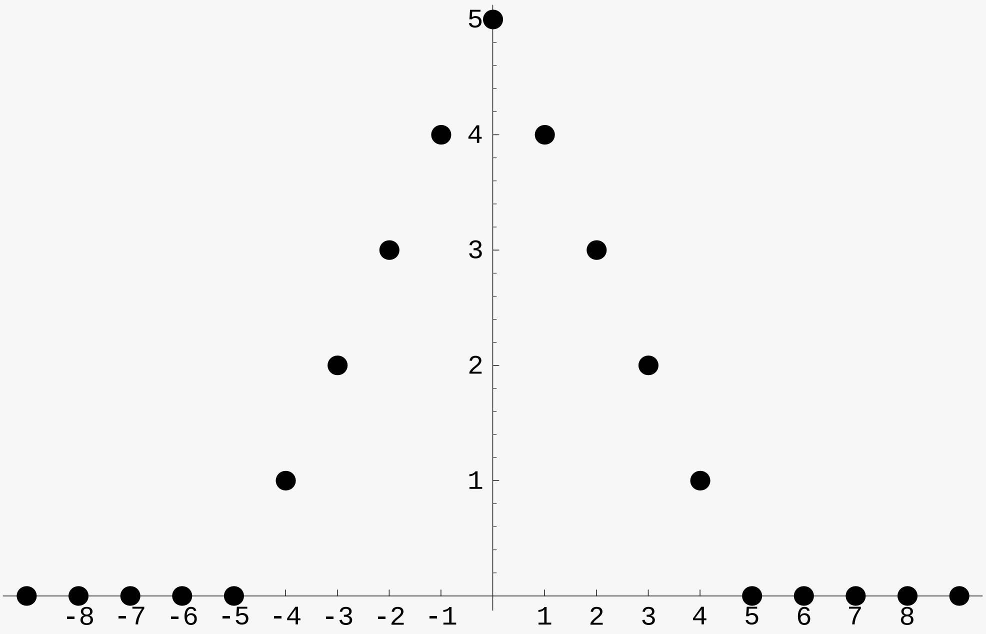
Then take the square root of the final result.

PROBLEM 4. Consider the following signal

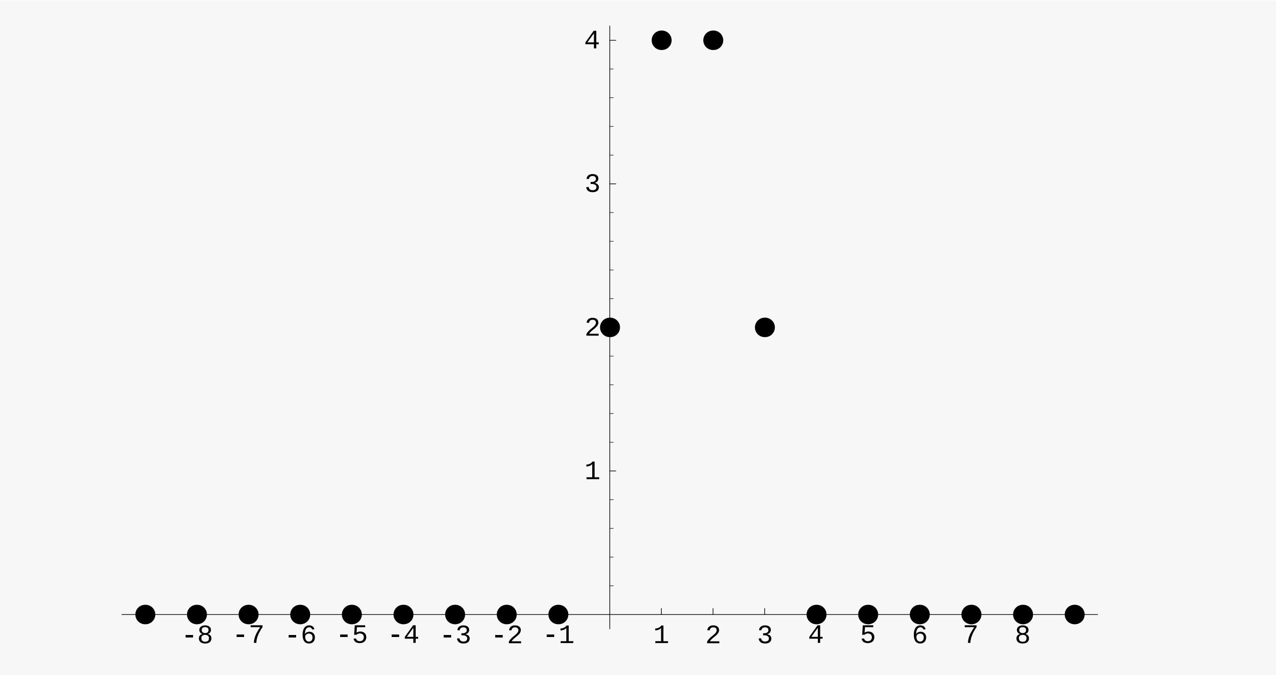
$$x[n] = (5 - |n|).(u[n + 5] - u[n - 6]).$$

Draw the following signals:

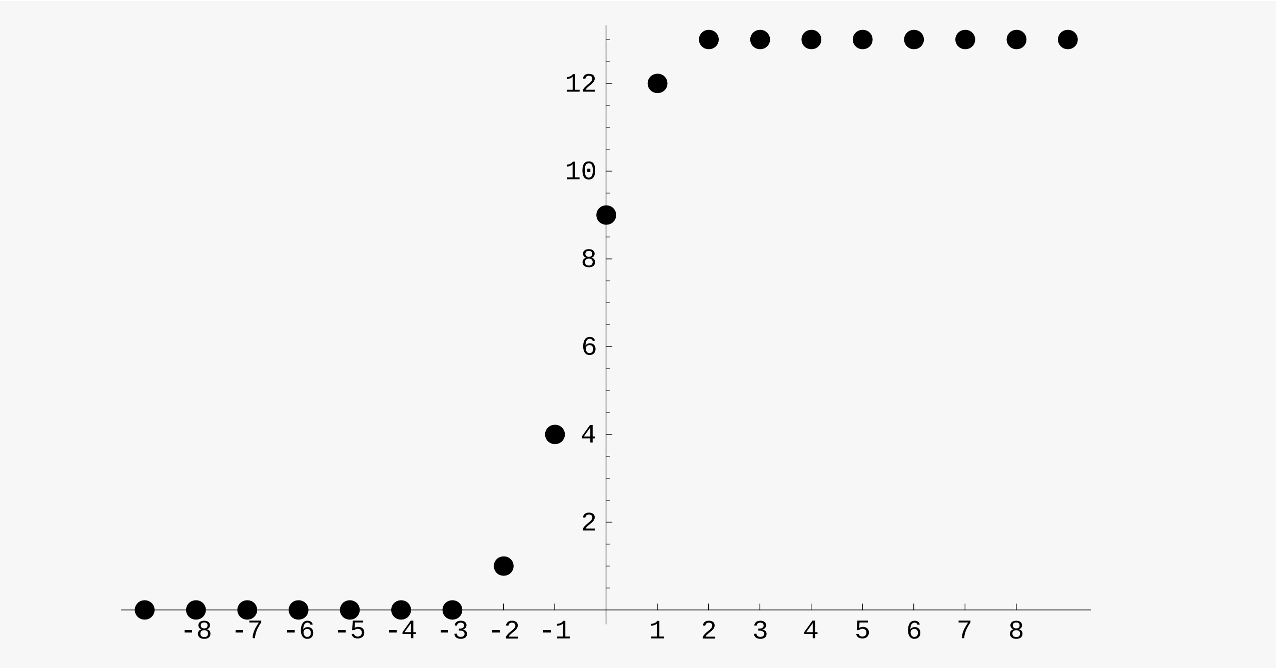
- $x[n]$.



- $x[-2n+3]$.



- $\sum_{k=-\infty}^{n} x[2k]$.



PROBLEM 5. Find the inverse z-transform of following series.

(a) $X(z) = \frac{1}{(1-1/4z^{-1})(1-1/2z^{-1})}$, $|z| > 1/2$.

(b) $X(z) = \frac{1}{(1-1/5z^{-1})(1+3z^{-1})}$, $3 > |z| > 1/5$.

(a) $X(z) = \frac{1}{(1-1/4z^{-1})(1-1/2z^{-1})}$, $|z| > 1/2$.
$X(z) = -\frac{1}{1-\frac{1}{4}z^{-1}} + \frac{2}{1-\frac{1}{2}z^{-1}}$. Both parts of X(z) are causal since the ROC is the region outside the farest pole ($\frac{1}{2}$), therefore $x[n] = -(\frac{1}{4})^n u[n] + 2(\frac{1}{2})^n u[n]$.

(b) $X(z) = \frac{1}{(1-1/5z^{-1})(1+3z^{-1})}$, $3 > |z| > 1/5$.
$X(z) = \frac{1/16}{1-\frac{1}{5}z^{-1}} + \frac{15/16}{1+3z^{-1}}$. The first part of $X(z)$ is causal and the second is anticausal, therefore $x[n] = \frac{1}{16}(\frac{1}{5})^n u[n] - \frac{15}{16}(-3)^n u[-n-1]$.