ÉCOLE POLYTECHNIQUE FÉDÉRALE DE LAUSANNE
School of Computer and Communication Sciences

**Handout 28** Signal Processing for Communications
Homework 11 Solution April 17, 2010

PROBLEM 1. When a random process $X[n]$ passes through the filter $h[n]$ to obtain a random process $Y[n]$, we will have:

$$r_y[n] = (h[n] * h[-n]) * r_x[n], \quad r_{xy}[n] = h[n] * r_x[n]$$

and in Fourier domain:

$$P_y(e^{j\omega}) = |H(e^{j\omega})|^2 P_x(e^{j\omega}), \quad P_{xy}(e^{j\omega}) = H(e^{j\omega}) P_x(e^{j\omega})$$

In this problem we have $h[n] = \delta[n] + \delta[n-1]$, so:

$$h[n] * h[-n] = \delta[n+1] + 2\delta[n] + \delta[n-1]$$

and since $r_x[n] = \sigma^2\delta[n]$, we can conclude:

$$r_y[n] = \sigma^2\delta[n+1] + 2\sigma^2\delta[n] + \sigma^2\delta[n-1]$$

PROBLEM 2. (a)

$$Y[n] = X[n] + \beta X[n-1] \rightarrow Y[n] = X[n] * h[n], \quad where \quad h[n] = \delta[n] + \beta\delta[n-1]$$

$$H(e^{j\omega}) = 1 + \beta e^{-j\omega} \rightarrow |H(e^{j\omega})|^2 = 1 + \beta^2 + 2\beta\cos(\omega)$$

We know that:

$$\begin{aligned}
\sum_{k=-\infty}^{+\infty} a^{|k|} e^{-j\omega k} &= 1 + \sum_{k=1}^{+\infty} (ae^{-j\omega})^k + \sum_{k=-\infty}^{-1} (a^{-1}e^{-j\omega})^k \\
&= 1 + \sum_{k=1}^{+\infty} (ae^{-j\omega})^k + (ae^{j\omega})^k \\
&= 1 + \frac{ae^{-j\omega}}{1 - ae^{-j\omega}} + \frac{ae^{j\omega}}{1 - ae^{j\omega}} \\
&= \frac{1 - a^2}{1 + a^2 - ae^{j\omega} - ae^{-j\omega}} \\
&= \frac{1 - a^2}{1 + a^2 - 2a\cos(\omega)}
\end{aligned}$$

since $R_x[k] = \delta^2 a^{|k|}$, we have:

$$P_x(e^{j\omega}) = \delta^2 \frac{1-a^2}{1+a^2-2a\cos(\omega)} \rightarrow P_y(e^{j\omega}) = \delta^2 \frac{1-a^2}{1+a^2-2a\cos(\omega)}(1+\beta^2+2\beta\cos(\omega))$$

(b) If $\beta = -a$, then $P_y(e^{j\omega}) = \delta^2(1-a^2)$ which is a constant value, so $Y[n]$ would be a white noise random process. In fact, in this case noise power is distributed uniformly on all frequencies.

PROBLEM 3. I call signal power and noise power $S$ and $N$ respectively. So, we will have:

$$S = \int_{-1}^{2} x^2 \; dx = \frac{x^3}{3} \; |_{-1}^{2} = \frac{8}{3} + \frac{1}{3} = 3$$

$$N = \int_{-1}^{0} (x-(-1))^2 \; dx + \int_{0}^{2} (x-1)^2 \; dx = \frac{1}{3} + \frac{2}{3} = 1$$

so signal to noise ratio would be equal to $\frac{S}{N} = 3$

prob 4.

a) $\int_0^1 (\frac{1}{2} + bx)\,dx = 1 \Rightarrow b = 1$

b) let $x_i = \frac{i}{2^r} = \frac{i}{K}$ for $i = 0, 1, \dots, K-1$

then $P(X = x_i) = \int_{x_i}^{x_{i+1}} (a + bx)\,dx$

$= a2^{-r} + b2^{-r} \frac{(x_i + x_{i+1})}{2} = a2^{-r} + b2^{-r} \times 2^{-r} \left(\frac{2i+1}{2}\right)$

c) we should look at the $x_i$s that fall in $[y, y+\delta]$. Let

$y_1 = \lceil y 2^r \rceil$

$y_2 = \lceil (y+\delta) 2^r \rceil \to y_1 - y_2 \approx \delta 2^r$

\# points in $[y, y+\delta] = \int_{x_{y_1}}^{x_{y_2}} (a + bx)\,dx$

$= a(x_{y_2} - x_{y_1}) + \frac{b}{2}(x_{y_2} - x_{y_1})(x_{y_2} + x_{y_1})$

$\overset{r \text{ is large}}{\cong} a\delta + \frac{b}{2}\delta(2y + \delta)$

$\simeq \delta(a + b(y + \frac{\delta}{2}))$