ÉCOLE POLYTECHNIQUE FÉDÉRALE DE LAUSANNE

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Handout 2 Signal Processing for Communications
Homework 1 March 1, 2010

PROBLEM 1. Decide whether the following signals are periodic, and if so, find the period.

(a) $x[n] = e^{j\frac{\pi}{\sqrt{2}}n}$.
A complex exponential of the form $e^{j\frac{2\pi M}{N}n}$ is periodic only if M,N $\in \mathbb{Z}$. Here M=1 and N=$2\sqrt{2}$, N $\notin \mathbb{Z}$ therefore $x[n]$ is not periodic.

(b) $x[n] = \frac{\sin(\pi n)}{\pi n}$.
$x[n] = \text{sinc}(n)$ which is not a periodic function.

(c) $x[n] = \sin(n)$.
$\sin(n) = 0$ only for $n = 0$ so the function has only one zero, and in order for this function to be periodic it has to come back to 0 if it has left it after some number of steps (period) therefore $\sin(n)$ is not periodic for n $\in \mathbb{Z}$. .

(d) $x[n] = 1 + \sin^2(\pi n)$.
Constant 1 does not affect periodicity, power 2 only tights the function and does not affect periodicity either. $\sin(\pi n)$ is 0 for every n $\in \mathbb{Z}$ so the period of $\sin(\pi n)$ is 1 therefore the period of $x[n]$ is 1 to.

(e) $x[n] = e^{j\frac{5\pi}{7}n} + e^{j\frac{3\pi}{4}n}$.
The reasoning is the same as in part a). $e^{j\frac{5\pi}{7}n} = e^{j\frac{2\pi 5}{14}n}$ is periodic since 5,14 $\in \mathbb{Z}$ of period T_1 =14 . $e^{j\frac{3\pi}{4}n} = e^{j\frac{2\pi 3}{8}n}$ is periodic since 3,8 $\in \mathbb{Z}$ of period T_2 =8. We know that the sum of two periodic functions is periodic of period = lcm(T_1,T_2). Therefore $x[n]$ is periodic of period 56.

PROBLEM 2. Compute the following sums.

(a) $S = \sum_{n=i}^{j} a^n$.
The sum is finite and $S = \sum_{n=0}^{j} a^n - \sum_{n=0}^{i-1} a^n = \frac{1-a^{j+1}}{1-a} - \frac{1-a^i}{1-a} = \frac{a^i - a^{j+1}}{1-a}$.

(b) $\sum_{n=1}^{\infty}(\frac{1}{2} + j\frac{\sqrt{3}}{2})^n$.
Such a sum converges if and only if the absolute value of the common ratio is less than one ($|r| < 1$). Its value can then be computed from the finite sum formulae $\sum_{k=0}^{\infty} r^k = \lim_{n\to\infty} \sum_{k=0}^{n} r^k = \lim_{n\to\infty} \frac{(1-r^{n+1})}{1-r} = \lim_{n\to\infty} \frac{1}{1-r} - \lim_{n\to\infty} \frac{r^{n+1}}{1-r}$. Since $r^{n+1} \to 0$ as $n \to \infty$ when $|r| < 1$ then $\sum_{k=0}^{\infty} r^k = \frac{1}{1-r}$.
The sum is infinite since $|\frac{1}{2} + j\frac{\sqrt{3}}{2}| = 1 \geq 1$.

(c) $S = \sum_{k=1}^{n} \sin(2\pi\frac{k}{N}), n \leq N$.
$S = \sum_{k=1}^{n} \frac{e^{2\pi\frac{k}{N}} - e^{-2\pi\frac{k}{N}}}{2j} = \sum_{k=1}^{n} \frac{1}{2j}(\frac{1-e^{2\pi\frac{n}{N}}}{1-e^{\frac{2\pi}{N}}} - 1 - \frac{1-e^{-2\pi\frac{n}{N}}}{1-e^{\frac{-2\pi}{N}}} + 1) =$
$\frac{\sin(\frac{2\pi}{N}) - (\sin\frac{2\pi n}{N}) + \sin(\frac{2\pi(n-1)}{N})}{2 - 2\cos(\frac{2\pi}{N})}$.

(d) $\sum_{n=1}^{\infty} e^{(1/2+j3/4)n}$.
The reasoning is the same as in b). The sum is infinite since $|e^{(1/2+j3/4)}| = e^{1/2} \geq 1$.

Problem 3. Compute the following integrals.

(a) $\int_0^{\infty} \frac{1}{1+x^4}\, dx$.
$\int_0^{\infty} \frac{1}{1+x^4}\, dx = \frac{1}{2}\int_{-\infty}^{\infty} \frac{1}{1+x^4}\, dx$ since $f(x) = \frac{1}{1+x^4}$ is pair.
$I = \frac{1}{2}\int_{-\infty}^{\infty} \frac{1}{1+x^4}\, dx = \frac{1}{2}\oint_C \frac{1}{1+z^4}\, dz$.

I is calculated using the residue theorem. The poles of $f(z) = \frac{1}{1+z^4}$ are $z_1 = e^{j\frac{\pi}{4}}, z_2 = e^{j\frac{3\pi}{4}}, z_3 = e^{j\frac{5\pi}{4}}, z_4 = e^{j\frac{7\pi}{4}}$. Only z_1 and z_2 are inside C and the integral over $C1$ is close to zero (it can be shown that the integral over $C1$ tends to zero as the radius of the arc tends to infinity) so $I = 2\pi j \sum_{z_1,z_2} Res\left[\frac{1}{1+z^4}\right]$.

$Resz_1 f(z) = \left|\frac{1}{\frac{d}{dz}(1+z^4)}\right| z_1 = \left|\frac{1}{4z^3}\right| z_1 = \frac{1}{4}e^{-j\frac{3\pi}{4}}$

$Resz_2 f(z) = \left|\frac{1}{4z^3}\right| z_2 = \frac{1}{4}e^{-j\frac{9\pi}{4}}$

$I = \frac{1}{2}2\pi j\frac{1}{4}(e^{-j\frac{3\pi}{4}} + e^{-j\frac{9\pi}{4}}) = \frac{\pi\sqrt{2}}{4}$

(b) $I = \int_{-\infty}^{\infty} \frac{\cos sx}{k^2+x^2}\, dx$.
$I = \int_{-\infty}^{\infty} \frac{\cos sx + j\sin sx}{k^2+x^2}\, dx$ since $\int_{-\infty}^{\infty} \frac{j\sin sx}{k^2+x^2}\, dx = 0$ as $\sin sx$ is symmetric according to the origin.
$I = \int_{-\infty}^{\infty} \frac{e^{jsx}}{k^2+x^2}\, dx = \oint_C \frac{e^{jsz}}{k^2+z^2}\, dz$. I is calculated using the residue theorem. The poles of $f(z) = \frac{e^{jsz}}{k^2+z^2}$ are $z_1 = jk$ and $z_2 = -jk$. Only z_1 is inside C the integral over $C1$ is close to zero (it can be shown that the integral over $C1$ tends to zero as the radius of the arc tends to infinity) so $I = 2\pi j Resz_1 f(z)$.
$Resz_1 f(z) = \left|\frac{e^{jsz}}{\frac{d}{dz}(k^2+z^2)}\right| z_1 = \frac{e^{-sk}}{2jk}$.
$I = \int_{-\infty}^{\infty} \frac{\cos sx}{k^2+x^2}\, dx = 2\pi j\frac{e^{-sk}}{2jk} = \frac{\pi}{k}e^{-sk}$.

(c) $I = \int_0^{2\pi} \frac{\sin\theta}{36-16\sin\theta}\, d\theta$.
$I = \int_0^{2\pi} \frac{1}{2j}\frac{(e^{j\theta}-e^{-j\theta})}{34-16\frac{(e^{j\theta}-e^{-j\theta})}{2j}}\, d\theta = \oint_C \frac{z-z^{-1}}{68j-16z+16z^{-1}}\frac{dz}{jz} = \oint_C \frac{z^2-1}{-16z^2+68jz+16}\frac{dz}{jz}$, change of variable $z = e^{j\theta}$ was used. C is $|z| = 1$.
I is calculated using the residue theorem. The poles of $f(z) = \frac{z^2-1}{-16jz^3-68z^2+16jz}$ are $z_1 = 0$, $z_2 = \frac{1}{4}j$, $z_3 = 4j$. Only z_1 and z_2 are inside C so $I = 2\pi j\sum_{z_1,z_2} Res\left[f(z)\right]$.

$Resz_1 f(z) = \left|\frac{z^2-1}{\frac{d}{dz}(-16jz^3-68z^2+16jz)}\right| z_1 = \left|\frac{z^2-1}{-48jz^2-136z+16j}\right| z_1 = -\frac{1}{16j} = \frac{j}{16}$.

$Resz_2 f(z) = \left|\frac{z^2-1}{-48jz^2-136z+16j}\right| z_2 = \frac{-\frac{1}{16}-1}{-15j} = -\frac{17j}{240}$.

$I = \int_0^{2\pi} \frac{\sin\theta}{36-16\sin\theta}\, d\theta = 2\pi j(\frac{j}{16} - \frac{17j}{240}) = \frac{\pi}{60}$.

Problem 4. Find the z-transform OF following series.

(a) $x[n] = a^n u[n]$.
$X(z) = \sum_{n=-\infty}^{\infty} x[n]z^{-n} = \sum_{n=0}^{\infty} a^n z^{-n} = \frac{1}{1-az^{-1}}$.
ROC: $|az^{-1}| < 1, |z| > |a|$.

(b) $x[n] = na^n u[n]$.
$X(z) = \sum_{n=-\infty}^{\infty} x[n]z^{-n} = \sum_{n=0}^{\infty} na^n z^{-n} = z\sum_{n=0}^{\infty} na^n z^{-n-1} = z\sum_{n=0}^{\infty} -\frac{d}{dz}(a^n z^{-n}) = -z\frac{d}{dz}(\sum_{n=0}^{\infty} a^n z^{-n}) = -z\frac{d}{dz}(\frac{1}{1-az^{-1}}) = \frac{az^{-1}}{(1-az^{-1})^2}$.
ROC: $|z| > |a|$.

PROBLEM 5. Find the inverse z-transform of following series.

(a) $X(z) = \frac{1}{(1-1/4z^{-1})(1-1/2z^{-1})}$, $|z| > 1/2$.
$X(z) = -\frac{1}{1-\frac{1}{4}z^{-1}} + \frac{2}{1-\frac{1}{2}z^{-1}}$. Both parts of X(z) are causal since the ROC is the region outside the farest pole ($\frac{1}{2}$), therefore $x[n] = -(\frac{1}{4})^n u[n] + 2(\frac{1}{2})^n u[n]$.

(b) $X(z) = \frac{1}{(1-1/5z^{-1})(1+3z^{-1})}$, $3 > |z| > 1/5$.
$X(z) = \frac{1/16}{1-\frac{1}{5}z^{-1}} + \frac{15/16}{1+3z^{-1}}$. The first part of $X(z)$ is causal and the second is anticausal, therefore $x[n] = \frac{1}{16}(\frac{1}{5})^n u[n] - \frac{15}{16}(-3)^n u[-n-1]$.