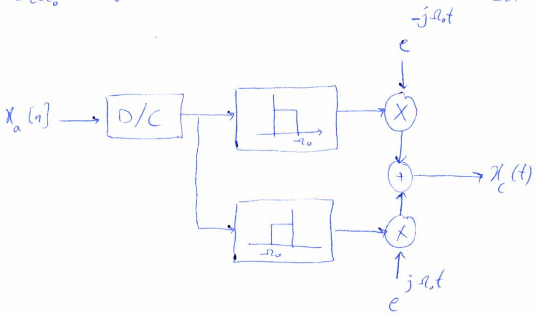
See next page.

PROBLEM 1-

1) We see that the maximum frequency is $2\Omega_0$, so we have to sample twice faster ($4\Omega_0$) $T_s =$

2)

3)



4. when we shift in frequency domain in order to prevent aliasing we need to be sure that the base band $2\Omega_0$ is large enough. That is

$$2(\Omega_1 - \Omega_0) \leq 2\Omega_0$$

$$\Rightarrow \Omega_1 \leq 2\Omega_0$$

If the base band ($2\Omega_0$) is not large enough or the bandwidth ($2(\Omega_1 - \Omega_0)$) is not small enough, then the smallest sampling frequency becomes the nyquist frequency.

$$\Omega_s = \begin{cases} 2(\Omega_1 - \Omega_0) & \text{if } \Omega_1 \leq 2\Omega_0 \\ 2\Omega_1 & \text{else} \end{cases}$$

Problem 2

1) $X_o(j\Omega) = \frac{1}{2\pi}\int_{-\infty}^{\infty} x_o(t)\, e^{-j\Omega t}\, dt = \frac{1}{2\pi}\sum_{m=-\infty}^{\infty} x[m]\int_{-\infty}^{\infty} rect(\tau) e^{-j\Omega \tau} e^{-j\Omega m} d$

$$= -\frac{1}{j\Omega 2\pi}\left(e^{-j\Omega/2} - e^{j\Omega/2}\right) X(e^{j\Omega})$$

$$= \frac{1}{2\pi} X(e^{j\Omega})\, sinc\left(\frac{\Omega}{2\pi}\right)$$

2) $x(t) = \sum_n x[n]\, sinc(t-n)$

$\rightarrow X(j\Omega) = \frac{1}{2\pi} X(e^{j\Omega})\, rect\left(\frac{\Omega}{2\pi}\right)$ $\left(sinc\left(\frac{t\,\Omega_N}{\pi}\right) \longleftrightarrow \frac{\pi}{\Omega_N}\, rect\left(\frac{\Omega}{2\Omega_N}\right)\right)$

The zero-order hod interpolation introduces a distortion ~~and~~ in the resulting signal. ~~because of the fact that sinc~~

And the interpolated signal is not band-limited in $[-\pi, \pi]$

now $x(t)$ can be obtained by

$$\begin{cases} x_o(t) = \sum_n x[n]\, rect(t-n) \rightarrow X_o(j\Omega) = X(e^{j\Omega})\, sinc(\Omega/2\pi) \\ x(t) = \sum_m x[m]\, sinc(t-m) \longrightarrow X(j\Omega) = X(e^{j\Omega})\, rect(\Omega/2\pi) \end{cases}$$

$\rightarrow x(t) = X_o(j\Omega)\, G(j\Omega) \Rightarrow G(j\Omega) = \dfrac{rect(\Omega/2\pi)}{sinc(\Omega/2\pi)}$

~~To avoid the distortions of x resulted from zero-holder~~
~~cutoff~~

Problem 3

1) $x(t) = \sum_{m=-\infty}^{\infty} x[m]\, I\left(\frac{t - mT_s}{T_s}\right)$ where $I(t) = \begin{cases} 1 - 2|t| & |t| < 1/2 \\ 0 & \text{o.w.} \end{cases}$

Assume $X(e^{j\omega}) = \begin{cases} 1 & |\omega| = \frac{2\pi}{3} \\ 0 & \text{o.w.} \end{cases}$

$I(t) = \text{rect}(2t) * \underbrace{\text{rect}(2t)}_{\text{fourier} = \frac{1}{2} \text{Sinc}\left(\frac{\Omega}{4\pi}\right)} \Rightarrow I(j\Omega) = \frac{1}{4} \text{Sinc}^2\left(\frac{\Omega}{4\pi}\right)$

$x(\ell t) \longleftrightarrow \frac{1}{\ell} X(j\Omega/\ell)$

2) $X(j\Omega) = \int_{-\infty}^{\infty} \sum_{m=-\infty}^{\infty} x[m]\, I\left(\frac{t - mT_s}{T_s}\right) e^{-j\Omega t} dt$

$= \sum x[n] e^{-jT_s\Omega m} \frac{1}{T_s} \int_{-\infty}^{\infty} I(z) e^{-jT_s\Omega z} dz =$

$= X(e^{jT_s\Omega}) \frac{1}{T_s} I(jT_s\Omega)$

$\Rightarrow X(j\Omega) = \begin{cases} \frac{1}{T_s} I(jT_s\Omega) & |\Omega| < \frac{2\pi}{3T_s} \\ 0 & \text{o.w.} \end{cases}$

Problem 4

$$\tilde{X}[k] = \mathrm{DFS}\,[\tilde{x}[m]] = \sum_{m=0}^{N-1} \tilde{x}[m]\, e^{-j\frac{2\pi}{N}mk}$$

$$\tilde{Y}[k] = \tilde{X}[2k] \qquad k \in \mathbb{Z} \;, \; N/2\text{-Periodic}$$

$$\Rightarrow \tilde{y}[m] = \frac{2}{N} \sum_{k=0}^{N/2-1} \tilde{Y}[k]\, e^{j\frac{2\pi}{N/2}mk} = \frac{2}{N} \sum_{l=0}^{N-1} \tilde{x}[l] \sum_{k=0}^{N/2-1} e^{-j\frac{2\pi}{N/2}(l-m)}$$

$$= \frac{2}{N} \sum_{l=0}^{N-1} \tilde{x}[l]\, \frac{N}{2}\, \delta[(l-m) \bmod N/2]$$

$$= \tilde{x}[m] + \tilde{x}[m+N/2] \qquad m=0,\dots,N/2-$$

So we have somehow aliasing but ~~this is~~ in the time domain.

PROBLEM 5-

Let $X[n]$ be a zero-mean white random process with auto correlation function

$r_x[n] = \sigma^2 \delta[n]$

we consider a leaky integrator $H(z) = \frac{1-\lambda}{1-\lambda z^{-1}}$

The impulse response is then $h[n] = (1-\lambda)\lambda^n u[n]$

$r_{xy}[n] = h[n] * r_x[n]$

$P_{xy}(e^{j\omega}) = H(e^{j\omega}) P_x(e^{j\omega})$

$P_x(e^{j\omega}) = \sum_{n=-\infty}^{+\infty} r_x[n] e^{-j\omega n} = \sum_{n=-\infty}^{+\infty} \sigma^2 \delta[n] e^{-j\omega n} = \sigma^2$

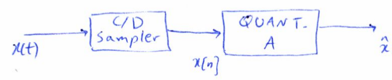
$P_{xy}(e^{j\omega}) = \sigma^2 \frac{1-\lambda}{1-\lambda e^{-j\omega}} \Longrightarrow r_{xy}[n] = \sigma^2 (1-\lambda)\lambda^n u[n]$

Thus : for $n \in [-3,-1]$, $n \in \mathbb{Z}$ $\quad r_{xy}[n] = 0$

for $n \in [0, 3]$, $n \in \mathbb{Z}$ $\quad r_{xy}[n] = \sigma^2 (1-\lambda)\lambda^n$

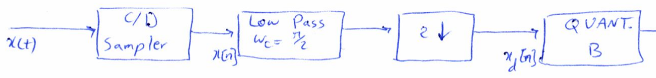
PROBLEM 6-

A:



$\hat{x}[n] = x[n] + e_A[n]$

B:



$\hat{x}_d[n] = x_d[n] + e_B[n]$

$MSE_A = e_A[n]$

$i_k = \frac{\hat{x}_{k-1} + \hat{x}_k}{2}$ and $\hat{x}_k = \frac{\int_{i_k}^{i_{k+1}} x f_x(x)dx}{\int_{i_k}^{i_{k+1}} f_x(x)dx}$

$MSE_B = e_B[n] + \int_{-1}^{1} |x(t) - x_d[n]|^2 dt$

min MSE_B in this case is min $e_B[n]$ + min $\int_{-1}^{1} |x(t) - x_d[n]|^2 dt$