Problem 1.

(a) We have

\[ A = \begin{bmatrix} 1 & 3 & 2 \\ 4 & 4 & 4 \\ 3 & 1 & 3 \end{bmatrix}, \]

and \( u = (2, 2, 0) \).

(b) It is \(-1\).

(c) Since the determinant is non-zero, the system can be solved uniquely.

(d), (e) The complexity of the Gaussian elimination is \( O(n^3) \).

Problem 2.

(a) Note that the first 4 entries of a codeword are the information bits. Thus the information vector used has been \( u = (1, 0, 1, 0) \) and the corresponding codeword to this information vector is \( (1, 0, 1, 0, 1, 0, 0) \). Thus the vector given in the question is not a codeword.

(b) There are 16 codewords. A list can be given by multiplying the matrix \( G \) with all the possible 4 dimensional information vectors \( u \) in the form of \( X = uG \).

(c) As mentioned in part (a), the first 4 bits of a codeword are the information bits. As a result, assuming \((u_1, u_2, u_3, u_4)\) has been the information vector used, we have \( u_3 = 0, u_4 = 1 \). To find \( u_1 \) and \( u_2 \) we should solve the system of equations \( \bar{u}\bar{G} = \bar{X} \), where \( \bar{u} = (u_1, u_2, 0, 1) \), \( \bar{X} = (0, 1, 0, 1) \) and \( \bar{G} \) is the matrix formed by deleting the first 4 columns of \( G \).

(d) \( C \) is the row space of the parity check matrix \( H \) corresponding to \( G \) (i.e., the matrix which has the property \( HG^T = 0 \)). \( H \) can is found to be

\[
H = \begin{bmatrix} 1 & 0 & 0 & 0 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 & 1 \end{bmatrix},
\]

(a) The space generated by the 3 rows of the matrix \( H \) given above is \( C^T \).
Problem 3.

(a) $r \times (2^r - 1)$

(b) Note that
\[
\begin{pmatrix}
1 \\
0 \\
\vdots \\
0
\end{pmatrix} +
\begin{pmatrix}
0 \\
1 \\
\vdots \\
1
\end{pmatrix} +
\begin{pmatrix}
1 \\
1 \\
\vdots \\
1
\end{pmatrix} = \mathbf{0}
\]

So, $d_{\text{min}} \leq 3$.

Furthermore, if $\exists i, j$ such that $V_i \oplus V_j = 0$, necessarily $V_i = V_j$, which is not possible by construction of $H$. Thus $d_{\text{min}} > 2$. Hence $d_{\text{min}} = 3$.

(c) $H_r X = 0$, and we are interested in the dimension of the kernel $H_r$.

$H_r$ is a $r \times (2^r - 1)$ matrix and it is full rank. Thus we can set any $2^r - 1 - r$ elements of the vector $X$ as desired and the rest of the elements will be determined by the system of equations $H_r X = 0$.

So, there are $2^{2r} - r - 1$ such vectors in the kernel of $H_r$ and thus its dimension is $2^r - r - 1$.

(d) To show that the code is linear, we should verify that

if $X_1 \in C$ and $X_2 \in C$ ⇒ $X_1 + X_2 \in C$

This is true because

if $X_1 \in C$ and $X_2 \in C$
⇒ $H_r X_1 = 0$ and $H_r X_2 = 0$
⇒ $H_r (X_1 + X_2) = H_r X_1 + H_r X_2 = 0$
⇒ $X_1 + X_2 \in C$.

So the code is linear.
Codeword length: $2^r - 1$.
Dimension of the code: $2^r - r - 1$.

$d_{\text{min}} = 3$.
Rate of the code $= \frac{k}{n} = \frac{2^r - r - 1}{2^r - 1}$.

Note that the rate of this code approaches 1 as $r \to \infty$. 