Problem 1.

(a) We have

$$A = \begin{bmatrix} 1 & 3 & 2 \\ 4 & 4 & 4 \\ 3 & 1 & 3 \end{bmatrix},$$

and $u = (2, 2, 0)$.

(b) It is -1.

(c) Since the determinant is non-zero, the system can be solved uniquely.

(d),(e) The complexity of the Gaussian elimination is $O(n^3)$.

Problem 2.

(a) Note that the first 4 entries of a codeword are the information bits. Thus the information vector used has been $u = (1, 0, 1, 0)$ and the corresponding codeword to this information vector is $(1, 0, 1, 0, 1, 0, 0)$. Thus the vector given in the question is not a codeword.

(b) There are 16 codewords. A list can be given by multiplying the matrix G with all the possible 4 dimensional information vectors u in the form of $X = uG$.

(c) As mentioned in part (a), the first 4 bits of a codeword are the information bits. As a result, assuming (u_1, u_2, u_3, u_4) has been the information vector used, we have $u_3 = 0$, $u_4 = 1$. To find u_1 and u_2 We should solve the system of equations $\bar{u}\bar{G} = X^T$, where $\bar{u} = (u_1, u_2, 0, 1)$, $\bar{X} = (0, 1, 0, 1)$ and $\bar{G}$ is the matrix formed by deleting the first 4 columns of G.

(d) C is the row space of the parity check matrix H corresponding to G (i.e., the matrix which has the property $HG^T = 0$). H can is found to be

$$H = \begin{bmatrix} 1 & 0 & 0 & 0 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 & 1 \end{bmatrix},$$

(a) The space generated by the 3 rows of the matrix Hgiven above is C^T.

Problem 3.

(a) $r \times (2^r - 1)$

(b) Note that

$$\begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ 1 \\ \vdots \\ 1 \end{pmatrix} + \begin{pmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{pmatrix} = \mathbf{0} \tag{1}$$

So, $d_{\min} \leq 3$.

Furthermore, if $\exists i, j$ such that $V_i \oplus V_j = 0$, necessarily $V_i = V_j$, which is not possible by construction of H. Thus $d_{min} > 2$. Hence $d_{\min} = 3$.

(c) $H_r X = 0$, and we are interested in the dimension of the kernel H.

H_r is a $r \times (2^r - 1)$ matrix and it is full rank. Thus we can set any $2^r - 1 - r$ elements of the vector X as desired and the rest of the elements will be determined by the system of equations $H_r X = 0$.

So, there are $2^{2^r - r - 1}$ such vectors in the kernel of H_r and thus its dimension is $2^r - r - 1$.

(d) To show that the code is linear, we should verify that

$$\text{if } X_1 \in C \text{ and } X_2 \in C \Rightarrow X_1 + X_2 \in C$$

This is true because

$$\begin{aligned} &\text{if } X_1 \in C \text{ and } X_2 \in C \\ &\Rightarrow H_r X_1 = 0 \text{ and } H_r X_2 = 0 \\ &\Rightarrow H_r(X_1 + X_2) = H_r X_1 + H_r X_2 = 0 \\ &\Rightarrow X_1 + X_2 \in C. \end{aligned}$$

So the code is linear.

Codeword length: $2^r - 1$.

Dimension of the code: $2^r - r - 1$.

$d_{min} = 3$.

Rate of the code $= \frac{k}{n} = \frac{2^r - r - 1}{2^r - 1}$.

Note that the rate of this code approaches 1 as $r \to \infty$.