Problem 1. 1. Since \(a \equiv a'; (\text{mod } m)\), it implies that \(a - a' = km\) for some integer \(k\). On the other hand we have:

\[
a' - a'' = (a - a')(a'^{-1} + a'^{-2}a' + \cdots + a'^{-n-1}) = mk(a'^{-1} + a'^{-2}a' + \cdots + a'^{-n-1})
\]

Therefore, \(a' - a''\) is divisible by \(m\), which implies that \(a' \equiv a''; (\text{mod } m)\).

2. The converse is not necessarily true. For example \((-1)^2 \equiv (+1)^2; (\text{mod 5})\) but \((-1) \neq (+1; (\text{mod 5})\).

Problem 2. We have

\[
a^3 + 3 = a^3 + 27 - 24 = a^3 + 3^3 - 24 = (a + 3)(a^2 - 3a + 9) - 24
\]

So, if \(a^3 + 3\) is divisible by \(a + 3\), \((a + 3)(a^2 - 3a + 9) - 24\) should be also divisible by \(a + 3\). Therefore 24 is divisible by \(a + 3\). But we know that the only factors of 24 are: 1, 2, 3, 4, 6, 8, 12, 24. This means that \(a + 3\) should be one of these numbers. Since \(a\) is assumed to be a positive integer number, the only possibilities for \(a\) are \(a = 1, 3, 5, 9, \) or 21.

Problem 3. Since \(n\) is assumed to be an odd number, we can write \(n = 2k + 1\)

1. We have:

\[
n^2 - 1 = (n - 1)(n + 1) = (2k + 1 - 1)(2k + 1 + 1) = (2k)(2k + 2) = 4k(k + 1).
\]

Notice that \(k\) and \(k + 1\) are two consecutive numbers. Therefore one of them is even and the other one is odd. In both cases, the product of them is an even number. So, \(k(k + 1) = 2m\). Therefore:

\[
n^2 - 1 = 4k(k + 1) = 4(2m) = 8m.
\]

This shows that \(n^2 - 1\) is a multiple of 8.

2. We will factor \(n^8 - 1\) as:

\[
n^8 - 1 = (n^4 - 1)(n^4 + 1) = (n^2 - 1)(n^2 + 1)(n^4 + 1).
\]

From the first part of the question we know that \(n^2 - 1\) is divisible by 8. So, \(n^2 - 1 = 8m\). Also, we know that the other factors are even numbers, since \(n\) is an odd number. Therefore we have:

\[
n^8 - 1 = (n^2 - 1)(n^2 + 1)(n^4 + 1) = 8m \times 2l \times 2p = 32mlp.
\]

This shows that \(n^8 - 1\) is divisible by 32.

Problem 4. First we write the congruence equation as

\[
7n \equiv -5; (\text{mod 2009})
\]

Now, we need to find \(7^{-1}(\text{mod 2009})\). Using Euclid’s algorithm, we can show that \(7^{-1} \equiv 287; (\text{mod 2009})\). Now, if we multiply both sides of the equation with 287 we have:

\[
287 \times 7n \equiv 287 \times (-5); (\text{mod 2009})
\]

Therefore:

\[
n \equiv -1435 \equiv 574; (\text{mod 2009})
\]