Problem 1

a)

b)

c)

d)
Problem 2

b) The system is causal because $h(n) = 0$ for $n < 0$. 
c)

\[(1 - r) \sum_{k=0}^{n} r^k = \sum_{k=0}^{n} r^k - r \sum_{k=0}^{n} r^k\]

\[= \sum_{k=0}^{n} r^k - \sum_{k=0}^{n} r^{k+1}\]

\[= \sum_{k=0}^{n} r^k - \sum_{k=1}^{n+1} r^k\]

\[= r^0 + \sum_{k=1}^{n} r^k - \left( r^{n+1} + \sum_{k=1}^{n} r^k \right)\]

\[= 1 - r^{n+1}\]

d) The system is stable because

\[\sum_{k=-\infty}^{\infty} |h(k)| = \sum_{k=0}^{\infty} 2^{-k}\]

\[= \lim_{k \to \infty} \frac{1 - 2^{-(k+1)}}{1 - \frac{1}{2}}\]

\[= 2 < \infty\]

e)

\[\frac{d}{dr} \frac{1 - r^{n+1}}{1 - r} = \frac{d}{dr} \sum_{k=0}^{n} r^k\]

\[= \sum_{k=0}^{n} \frac{d}{dr} r^k\]

\[= \sum_{k=0}^{n} kr^{k-1}\]

\[= r^{-1} \sum_{k=0}^{n} kr^k\]

f)

\[y(n) = \sum_{k=-\infty}^{\infty} h(k) x(n - k)\]

\[= \sum_{k=0}^{\infty} 2^{-k} (n - k)\]

\[= n \sum_{k=0}^{\infty} 2^{-k} - \sum_{k=0}^{\infty} 2^{-k}\]

\[= 2n - 2 \left. \frac{d}{dr} \left( \frac{1}{1 - r} \right) \right|_{r=2}\]

\[= 2n - \frac{1}{2} \left( \frac{1}{(1 - 1/2)^2} \right)\]

\[= 2 (n - 1)\]
Problem 3

We have a system from which we only know a linear time-invariant recurrence relation between the input \( x(n) \) and the output \( y(n) \), namely

\[
y(n + 1) = y(n) + x(n)
\]

\[
\lim_{m \to -\infty} y(m) = 0
\]

a) \( y(4) = y(3) + x(3) = y(2) + x(2) + x(3) = y(1) + x(1) + x(2) + x(3) = y(0) + x(0) + x(1) + x(2) + x(3) \)

b) Since \( \lim_{m \to -\infty} y(m) = 0 \)

\[
y(n) = \lim_{m \to -\infty} y(n)
\]

\[
= \lim_{m \to -\infty} \left( y(m) + \sum_{k=m}^{n-1} x(k) \right)
\]

\[
= \sum_{k=-\infty}^{n-1} x(k)
\]

The output signal is a summation of the input signal.

c) For \( n < 0 \), since \( \delta(n) = 0 \) then \( h(n + 1) = h(n) \) and \( h(n + 1) = 0 \) if \( n < 0 \). \( h(1) = \delta(0) = 1 \) and for \( n \geq 1 \) since \( \delta(n) = 0 \) then \( h(n + 1) = h(n) = h(n - 1) = \ldots = h(1) = 1 \). We finally found that

\[
h(n) = \begin{cases} 
1 & \text{if } n \geq 1 \\
0 & \text{otherwise}
\end{cases}
\]

d) The general output is

\[
y(t) = \sum_{k=-\infty}^{\infty} h(k) x(n - k)
\]

\[
= \sum_{k=1}^{\infty} x(n - k)
\]

\[
= \sum_{k=-\infty}^{-1} x(n + k)
\]

\[
= \sum_{k=-\infty}^{n-1} x(k)
\]

This is same answer than the one found in b)

e) The system is unstable because

\[
\sum_{k=-\infty}^{\infty} |h(k)| = \sum_{k=1}^{\infty} 1 = \infty.
\]

An example of unstable input signal is \( x(n) \equiv 1 \).

Problem 4

\[
\sin(10\pi t) + \sin(30\pi t)
\]

a) The maximum frequency between these two sinusoïds is 15Hz. So \( f_s > 30Hz \).
b) If we use the sampling frequency \( f_s = 20 \text{Hz} \), what the sampled signal would be?

\[
\sin \left(10\pi \frac{n}{20}\right) + \sin \left(30\pi \frac{n}{20}\right) = \sin \left(\frac{\pi}{2} n\right) + \sin \left(\frac{3\pi}{2} n\right)
\]
\[
= \sin \left(\frac{\pi}{2} n\right) + \sin \left(-\frac{\pi}{2} n\right)
\]
\[
= \sin \left(\frac{\pi}{2} n\right) - \sin \left(\frac{\pi}{2} n\right)
\]
\[
= 0
\]

c) The ideal filter will suppress all frequency greater than \( \frac{f_s}{2} \). So the sampled signal would be

\[
\sin \left(10\pi \frac{n}{20}\right) = \sin \left(\frac{\pi}{2} n\right)
\]

d) The sampling frequency is \( f_s = 25 \text{Hz} \) and we use an ideal interpolator. What the reconstructed signal would be? The first sinusoid is perfectly reconstructed because \( 5 \text{Hz} < \frac{25}{2} \text{Hz} \), but the second one is altered. An alias frequency for \( 15 \text{Hz} \) and lying in the interval \((-25/2, 25/2)\) is given by \( 15 - 25 = -10 \text{Hz} \). Thus the reconstructed signal would be

\[
\sin (10\pi t) - \sin (20\pi t)
\]

**Problem 5**

We have a continuous signal \( x(t) \) with a missing part between \( t = 1 \) and \( t = -1 \). The signal is zero everywhere and the only information we know in the missing part is \( x(0) = 1 \). We want to interpolate this signal.

a) We have to solve two linear system of equation, one for \( \{x(-1), x(0)\} \) and an other for \( \{x(0), x(1)\} \).

For the first part we have

\[
\begin{cases}
-a + b = 0 \\
b = 1
\end{cases}
\]

this gives \( a = 1 \) and \( b = 1 \). For the second part we have

\[
\begin{cases}
a + b = 0 \\
b = 1
\end{cases}
\]

and we have \( a = -1 \) and \( b = 1 \).

b) We have to solve the following linear system of equation

\[
\begin{cases}
a - b + c = 0 \\
c = 1 \\
a + b + c = 0
\end{cases}
\]

we find \( a = -1, b = 0 \) and \( c = 1 \).

c) Now we have two equations given by the derivatives i.e. \( 3at^2 + 2bt + c \). The system is

\[
\begin{cases}
-a + b - c + d = 0 \\
d = 1 \\
3a - 2b + c = 0 \\
c = 1
\end{cases}
\]

and we find \( a = -2, b = -3, c = 0, d = 1 \)