Random matrix theory: lecture 8

Rate-diversity tradeoff in multi-antenna channels

Back to the third scenario: the quantity of interest is

$$\mathbb{P}_{out}(R) = \min_{Q \geq 0 : \operatorname{Tr} Q \leq P} \mathbb{P}\left(\log\det(I + HQH^*) < R\right)$$

We again assume that h_{jk} are iid $\sim N_{\mathbb{C}}(0,1)$, and would like to perform the analysis of $\mathbb{P}_{out}$ in the high SNR regime ($P \to \infty$). [ref: Zheng-Tse 03]

In the ergodic case, we have seen that

$$C = \max_{Q \geq 0 : \operatorname{Tr} Q \leq P} \mathbb{E}\left(\log\det(I + HQH^*)\right) = \mathbb{E}\left(\log\det(I + \tfrac{P}{n} HH^*)\right)$$

$$= \mathbb{E}\left(\sum_{j=1}^{n} \log(1 + P\lambda_j)\right) \approx \min(m,n) \cdot \log P \quad \text{as } P \to \infty$$

(as shown by a simple heuristics)

Let us therefore choose a target rate $R = r \log P$, for some $0 \leq r \leq \min(m,n)$. We expect that

$$\mathbb{P}_{out}(r \log P) \approx P^{-d(r)} \quad \text{for some } d(r) > 0.$$

$\Rightarrow$ definition: the diversity order $d(r)$ associated to a rate r is: $d(r) := \lim_{P \to \infty} - \dfrac{\log(\mathbb{P}_{out}(r \log P))}{\log P}$

[= upper bound on the diversity order of the error probability arising from a given coding scheme]

Our aim now is to compute $d(r)$ for

H $m \times n$ matrix with iid $\sim N_{\mathbb{C}}(0,1)$ entries and $m \geq n$.

[without loss of generality]

Notation: $f(P) \doteq g(P)$ if $\lim_{P \to \infty} \frac{\log f(P)}{\log P} = \lim_{P \to \infty} \frac{\log g(P)}{\log P}$

[same "diversity order" for f & g]

Lemma

× $\quad \mathbb{P}_{out}(r \log P) \doteq \mathbb{P}\left(\log\det(I + P H^* H) < r \log P\right)$

Proof

- $A(P) := \mathbb{P}(\underbrace{\log\det(I + \frac{P}{n} H^* H))}_{< r \log P} \overset{\text{choose } Q = \frac{P}{n} I}{\geq} \mathbb{P}_{out}(r \log P)$

× • for any $Q \geq 0$ s.t. $\operatorname{Tr} Q \leq P$, we have $Q \leq P\, I$;

× $\quad$ since $A \mapsto \log\det A$ is increasing on the set of positive definite matrices, this implies

$\log\det(I + H Q H^*) \leq \log\det(I + P H^* H)$ for any Q,

so $\mathbb{P}_{out}(r\log P) \geq \mathbb{P}\left(\log\det(I + P H^* H) < r \log P\right) := B(P)$

- finally, $A(P) \doteq B(P)$; indeed:

$B(P) \leq A(P) = \mathbb{P}\left(\log\det(I + \frac{P}{n} H^* H) < r \log P\right)$

$\doteq \mathbb{P}\left(\log\det(I + P H^* H) < r \log(nP)\right)$

$\doteq \mathbb{P}\left(\log\det(I + P H^* H) < r \log P\right) = B(P)$ #

Bottom line: the hard optimization problem is gone!

Therefore,

$$\mathbb{P}_{out}(r \log P) \doteq \mathbb{P}\Big(\sum_{j=1}^{n} \log(1 + P \overset{\text{ev. of } H^*H}{\downarrow}{\lambda_j}) < r \log P\Big)$$

$$= \int_{D_\lambda(r)} p(\lambda_1 \dots \lambda_n)\, d\lambda_1 \dots d\lambda_n$$

where $p(\lambda_1, \dots, \lambda_n) = C_n \cdot \exp\big(-\sum_{j=1}^{n} \lambda_j + (m-n) \sum_{j=1}^{n} \log \lambda_j\big)$

$$\cdot \prod_{j<k} (\lambda_k - \lambda_j)^2 \quad \text{(see lecture 3)}$$

and $D_\lambda(r) = \Big\{ \underbrace{0 \le \lambda_1 \le \dots \le \lambda_n}_{\text{eigenvalues ordered in ascending order}} : \sum_{j=1}^{n} \log(1 + P\lambda_j) < r \log P \Big\}$

Let us now make the following <u>change of variables</u>:

$$\underset{\ge 0}{\lambda_j} = P^{-\overset{\in \mathbb{R}}{\alpha_j}} = e^{-\alpha_j \log P} \quad \Rightarrow d\lambda_j = -(\log P)\, e^{-\alpha_j \log P} d\alpha_j$$

so

$$\mathbb{P}_{out}(r \log P) \doteq \int_{D_\alpha(r)} q(\alpha_1 \dots \alpha_n)\, d\alpha_1 \dots d\alpha_n$$

where $q(\alpha_1 \dots q_n) = C_n \exp\big(-\sum_{j=1}^{n} P^{-\alpha_j} - (m-n) \sum_{j=1}^{n} \alpha_j \log P\big)$

$$\cdot \prod_{j<k} (P^{-\alpha_k} - P^{-\alpha_j})^2 (\log P)^n \exp\big(-\sum_{j=1}^{n} \alpha_j \log P\big)$$

and $D_\alpha(r) = \Big\{ \underset{(\alpha_j \in \mathbb{R})}{\alpha_1 \ge \dots \ge \alpha_n} : \sum_{j=1}^{n} \log(1 + P^{1-\alpha_j}) < r \log P \Big\}$

At this point, let us make a couple of observations:

a) as $P \to \infty$, $\exp(-P^{-\alpha_j}) \begin{cases} \text{decays super-polynomially to } 0 & \text{if } \alpha_j < 0 \\ \text{tends to } 1 & \text{if } \alpha_j \geq 0 \end{cases}$

so we may restrict the integral to $\alpha_j \geq 0 \ \forall j$

b) as $P \to \infty$, $\log(1 + P^{1-\alpha_j}) \approx \begin{cases} (1-\alpha_j) \log P & \text{if } \alpha_j \leq 1 \\ 0 & \text{if } \alpha_j > 1 \end{cases}$

so $\log(1+P^{1-\alpha_j}) \approx \underbrace{(1-\alpha_j)^+}_{\text{positive part}} \log P$

We may therefore replace the domain of integration $D_\alpha(r)$ by $\widetilde{D}_\alpha(r) = \left\{ \alpha_1 \geq \ldots \geq \alpha_n \geq 0 : \sum_{j=1}^{n} (1-\alpha_j)^+ < r \right\}$

So $\mathbb{P}_{out}(r \log P) \doteq \int_{\widetilde{D}_\alpha(r)} \widetilde{q}(\alpha_1, \ldots, \alpha_n)\, d\alpha_1 \ldots d\alpha_n$

where $\widetilde{q}(\alpha_1 \ldots \alpha_n) = C_n (\log P)^n e^{-(m-n+1)\sum_j \alpha_j \log P}$

$$\cdot \prod_{j<k} \left(P^{-\alpha_k} - P^{-\alpha_j} \right)^2$$

Now, since $\alpha_1 \geq \ldots \geq \alpha_n$, we have

$$\prod_{j<k} \left(P^{-\alpha_k} - P^{-\alpha_j} \right)^2 \approx \prod_{j<k} P^{-2\alpha_k} = \prod_k P^{-2(k-1)\alpha_k}$$

$$= e^{-2\sum_k (k-1)\alpha_k \log P}$$

Note also that in terms of "diversity order", $(\log P)^n$ is equivalent to a constant, since $\lim_{P \to \infty} \frac{\log((\log P)^n)}{\log P} = 0$ (which is in turn equivalent to 1)

Finally,

$$P_{out}(r \log P) \doteq \int_{\tilde{D}_\alpha(r)} e^{-\overbrace{\sum_j (m-n+2j-1)\,\alpha_j \log P}^{:=f(\alpha)}} d\alpha_1 \ldots d\alpha_n$$

$$= \int_{\tilde{D}_\alpha(r)} P^{-f(\alpha)} d\alpha_1 \ldots d\alpha_n \underset{\uparrow}{\doteq} P^{-\min_{\tilde{D}_\alpha(r)} f(\alpha)}$$

Laplace integration method

The final answer is therefore : $P_{out}(r \log P) \doteq P^{-d(r)}$, where

$$\boxed{d(r) = \min_{\tilde{D}_\alpha(r)} f(\alpha) = \min_{\{\alpha_1 \geq \ldots \geq \alpha_n \geq 0 : \sum_{j=1}^n (1-\alpha_j)^+ < r\}} \sum_{j=1}^n (m-n+2j-1)\,\alpha_j}$$

<u>Ex:</u> <u>$m = n = 2$</u>

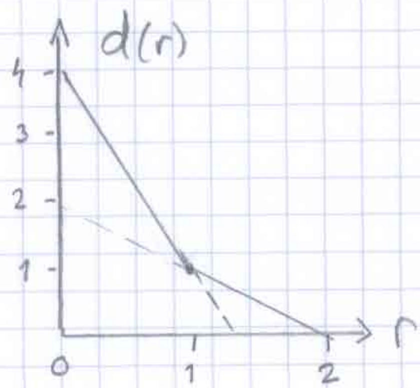
$$d(r) = \min_{\{\alpha_1 \geq \alpha_2 \geq 0 : (1-\alpha_1)^+ + (1-\alpha_2)^+ < r\}} \alpha_1 + 3\alpha_2$$

<u>$0 < r < 1$</u> : $\alpha_1 = 1$ & $\alpha_2 = 1 - r$ & $d(r) = 4 - 3r$

i.e. $\lambda_1 \sim P^{-1}$ $\qquad$ $\lambda_2 \sim P^{r-1}$

<u>$1 < r < 2$</u> : $\alpha_1 = 2 - r$ & $\alpha_2 = 0$ & $d(r) = 2 - r$

i.e. $\lambda_1 \sim P^{r-2}$ $\qquad$ $\lambda_2 \sim 1$



<u>tradeoff</u> :

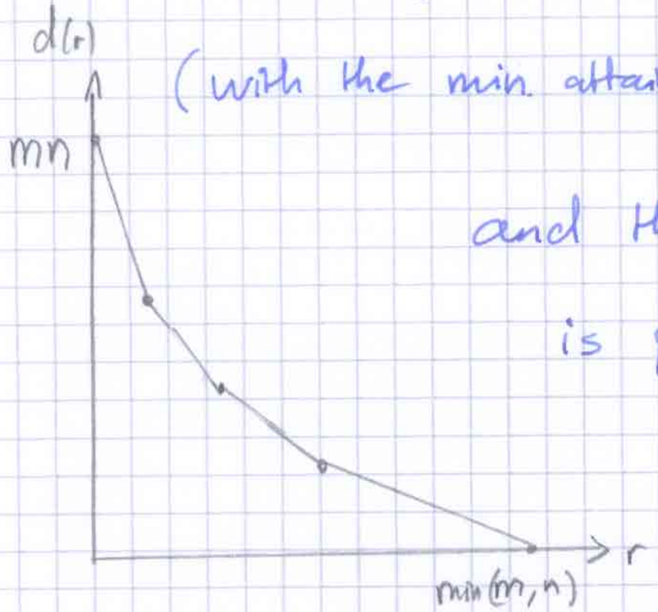
$$\begin{cases} r \sim 2 : \text{ full rate, low diversity} \\ r \sim 0 : \text{ low rate, full diversity} \end{cases}$$

Picture in the general case

It turns out that for integer $r = k$, the solution of the above minimization problem is:

$$d(k) = (m-k)(n-k)$$

(with the min attained at $\alpha_1 = \dots = \alpha_{n-k} = 1$, $\alpha_{n-k+1} = \dots = \alpha_n = 0$)



and the function inbetween is piecewise linear (& convex)

Interpretation

With m receive antennas and n transmit antennas, a rate $k \log P$ can be "achieved" if $k \leq \min(m,n)$; k antennas on each side contribute to this rate, while the remaining $m-k$ and $n-k$ antennas provide diversity.

<u>NB</u>: for a target rate $R = k \log P$, the outage event is the event that the channel matrix H gives rise to less than k independent reliable scalar channels.