Random matrix theory: Lecture 21

Gaussian random matrices and free probability

We have already seen that the following result holds (lecture 16):

- Let $A^{(n)} = \text{diag}(a_1, \ldots, a_n)$, with $a_j \in \mathbb{R}$ (deterministic),
  
  be such that $F_n^A(t) = \frac{1}{n} \# \{ 1 \leq j \leq n: a_j \leq t \} \xrightarrow{n \to \infty} F^A(t)$,

  with corresponding Stieltjes transform $g_A(z)$.

- Let $H$ be a real symmetric matrix with iid $\sim \mathcal{N}(0,1)$ entries in the upper triangular part, and $H^{(\infty)} = \frac{1}{\sqrt{n}} H$.

- Let $B^{(n)} := A^{(n)} + H^{(\infty)}$ and $\lambda_j^{(n)}$ be the ev of $B^{(n)}$. Then
  
  $F_n^B(t) = \frac{1}{n} \# \{ j: \lambda_j^{(n)} \leq t \} \xrightarrow{n \to \infty} F^B(t)$ a.s.

  whose Stieltjes transform $g_B(z)$ satisfies $g_B(z) = g_A(z + g_H(z))$.

Generalizations of this result:

- The result still holds if $A^{(n)}$ is real symmetric with eigenvalues $\lambda_1^{(n)}, \ldots, \lambda_n^{(n)}$ and
  
  $F_n^A(t) = \frac{1}{n} \# \{ j: \lambda_j^{(n)} \leq t \} \xrightarrow{n \to \infty} F^A(t)$

  $F_n^B(t)$

Proof:

- Let $O^{(n)}$ orthogonal and $D^{(n)} = \text{diag}(a_1^{(n)}, \ldots, a_n^{(n)})$ s.t. $A^{(n)} = O^{(n)}D^{(n)}O^{(n)\top}$

  $B^{(n)} = O^{(n)}(D^{(n)} + H^{(n)}H^{(n)\top}O^{(n)})O^{(n)\top}$

  same dist. as $H^{(n)}$

  same e.v. as $B^{(n)}$

  $\Rightarrow$ see lecture 2
The result still holds if $A^{(n)}$ is random and independent of $H^{(n)}$ with $F_{n}^{A}(t) \rightarrow F^{A}(t)$ a.s.

**Proof:**

Conditioned on $A^{(n)}$, the result holds since $F_{n}^{A}(t) \rightarrow F^{A}(t)$ a.s. and $F^{A}$ is deterministic.

**Remark:**

The result still holds for $H$ with non-Gaussian entries, i.e., non-orthogonally invariant, but this requires further work.

In lecture 16, we have also seen a result of the same flavor:

- Let $A^{(n)}$ be real symmetric & independent of $H$ such that $F_{n}^{A}(t) \rightarrow F^{A}(t)$ a.s. with Stieltjes transform $g_{A}(z)$.
- Let $H$ be $n \times n$ with iid $\sim N_{\mathbb{R}}(0,1)$ entries and $W^{(n)} = \frac{1}{n} HHT$.
- Let $B^{(n)} = A^{(n)} + W^{(n)}$. Then $F_{n}^{B}(t) \rightarrow F^{B}(t)$ a.s., whose Stieltjes transform $g_{B}(z)$ satisfies

\[
g_{B}(z) = g_{A}(z - \frac{1}{1 + g_{B}(z)})
\]
Question

Is there a general rule for computing the limiting eigenvalue distribution of the sum of two independent random matrices $A^{(n)} + B^{(n)}$?

Answer

- A particular case of independent random matrices are deterministic matrices; and in this case, we know that there is no simple rule for computing the eigenvalues of $A^{(n)} + B^{(n)}$ from the eigenvalues of $A^{(n)}$ and $B^{(n)}$ separately, mainly because of the fact that they do not share the same eigenvectors in general.

- Moreover, even in the case where $A^{(n)}$ and $B^{(n)}$ share the same eigenvectors (when they are both diagonal, or both circulant, e.g.), everything is possible regarding the limiting eigenvalue distribution of $A^{(n)} + B^{(n)}$. 
Example:

Let \( A^{(n)} = \text{diag}(\frac{1}{n}, \frac{2}{n}, \ldots, \frac{n-1}{n}, \frac{n}{n}) = \mathbf{B}^{(n)} \).

Then the limiting eigenvalue distribution of both \( A^{(n)} \) and \( \mathbf{B}^{(n)} \) is the uniform distribution on \([0, 1]\).

Also, \( A^{(n)} + \mathbf{B}^{(n)} = \text{diag}(\frac{2}{n}, \frac{4}{n}, \ldots, \frac{2n}{n}) \) has the limiting eigenvalue distribution the uniform distribution on \([0, 2]\).

Let now \( \tilde{\mathbf{B}}^{(n)} = \text{diag}(\frac{1}{n}, \frac{n-1}{n}, \ldots, \frac{2}{n}, \frac{1}{n}) \).

Then the limiting eigenvalue distribution of \( \tilde{\mathbf{B}}^{(n)} \) is also the uniform distribution on \([0, 1]\), but \( A^{(n)} + \tilde{\mathbf{B}}^{(n)} = \text{diag}(\frac{n+1}{n}, \frac{n+1}{n}, \ldots, \frac{n+1}{n}) \) has for limiting eigenvalue distribution the Dirac distribution at point \( x = 1 \).

In order to find a general rule for the limiting eigenvalue distribution of sums of random matrices, we need therefore to find a more restrictive condition than the independence of \( A^{(n)} \) and \( \mathbf{B}^{(n)} \).
Important observation

When dealing with distributions of (eigenvalues of) random matrices, the framework of classical probability is not the best one, since any two classical random variables $X$ and $Y$ commute: $XY = YX$, but the same is not true for random matrices.

$\Rightarrow$ Non-commutative probability

Let $A$ be the set of $n \times n$ matrices; $A$ is a non-commutative algebra, with addition $A + B$, multiplication $A \cdot B$ and unit element $A = I$. (*)

Def: an expectation on $A$ is an application $\varphi: A \to \mathbb{R}$ st.

- $\varphi$ is linear: $\varphi(A + cB) = \varphi(A) + c \varphi(B)$
- $\varphi(I) = 1$
- $\varphi(A) \geq 0$ if $A \geq 0$

Examples:

- $\varphi(A) := \frac{1}{n} \text{Tr} A$
- $\varphi(A) := a_{ii}$

(*) and matrices are called non-commutative random variables.
Remarks:

- The set of classical random variables also forms an algebra, which is moreover commutative.
- So far, non-commutative random variables are non-deterministic matrices (random matrices will come later).

What is the distribution of a non-commutative r.v.?

- The "distribution" of a matrix $A$ is defined through its moments: $M_k = \varphi(A^k), \quad k \geq 0$,
  but there is in general no corresponding classical distribution $\mu_A$ on $\mathbb{R}$.

- For $\varphi(A) = \frac{1}{n} \text{Tr} A$, there is:

$$
\mu_A = \frac{1}{n} \sum_{j=1}^{\infty} Y_j^A,
$$
where $Y_j^A$ is c.v. of $A$

$$
M_k^A = \sum_{j=1}^{\infty} \left( \frac{1}{j} \right)^k = \frac{1}{n} \sum_{j=1}^{\infty} (Y_j^A)^k = \frac{1}{n} \text{Tr}(A^k) = \varphi(A^k).
$$

- $\varphi_A(z) = \frac{1}{n} \sum_{j=1}^{\infty} \frac{1}{j} \text{Tr}(A^j)^{-1} = \varphi((A - zI)^{-1}), \quad z \in \mathbb{C} \setminus \mathbb{R}.$

- Note moreover that in this case, we have

$$
\varphi(AB) = \frac{1}{n} \text{Tr}(AB) = \frac{1}{n} \text{Tr}(BA) = \varphi(BA).
$$