Random matrix theory: lecture 19

Non-negative and positive definite matrices [ref: Horn-Johnson chap. 7]

× Def: • an $n \times n$ matrix A is non-negative definite

× if $x^* A x \geq 0 \quad \forall x \in \mathbb{C}^n$; notation: $A \geq 0$

• A is moreover positive definite if $x^* A x > 0 \quad \forall x \neq 0$;

notation: $A > 0$

× Properties:

• $A \geq 0 \implies A = A^*$

• $A \geq 0 \overset{(*)}{\implies} A = U \Lambda U^*$ with U $n \times n$ unitary

and $\Lambda = \mathrm{diag}(\lambda_1 \ldots \lambda_n)$ with $\lambda_j \geq 0$

• $A, B \geq 0$, $\underset{(\in \mathbb{R})}{\alpha, \beta \geq 0} \implies \alpha A + \beta B \geq 0$

• $A \geq 0$ and $\mathrm{rank}\, A = m \leq n \overset{(*)}{\implies} \exists\, V$ $n \times m$ matrix s.t. $A = V V^*$

Pf: $A = U \Lambda U^* = \sum_{j=1}^{n} \lambda_j u_j u_j^*$ with $\lambda_1 \ldots \lambda_m > 0$, $\lambda_{m+1} = \ldots = \lambda_n = 0$

define $v_j = \frac{1}{\sqrt{\lambda_j}} u_j \leftarrow$ (j^{th} column of U) for $1 \leq j \leq m$

then $A = \sum_{j=1}^{m} \underbrace{v_j v_j^*}_{\text{rank one } n \times n \text{ matrices}} = V V^*$ where $V = (v_1 | \ldots | v_m)$

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• $A > 0$ iff $A \geq 0$ and A is invertible

iff $A \geq 0$ and $\mathrm{rank}\, A = n$

iff $A = V V^*$ with V $n \times n$ invertible

(*) NB: the reverse implications are clearly also true!

Def: Schur or Hadamard product:

$$(A \circ B)_{jk} := a_{jk} b_{jk} \qquad A, B \ n \times n \text{ matrices}$$

Property: $A, B \geq 0 \implies A \circ B \geq 0$

Pf: $A = VV^*$ for some $n \times m$ V, $B = WW^*$ for some $n \times p$ W

i.e. $a_{jk} = \sum_{\ell=1}^{m} V_{j\ell} \overline{V_{k\ell}}$, $b_{jk} = \sum_{\ell'=1}^{p} W_{j\ell'} \overline{W_{k\ell'}}$

so $(A \circ B)_{jk} = \sum_{\ell,\ell'=1}^{m,p} \underbrace{(V_{j\ell} W_{j\ell'})}_{= y_j^{(\ell,\ell')}} \underbrace{\overline{(V_{k\ell} W_{k\ell'})}}_{= \overline{y_k^{(\ell,\ell')}}}$

i.e. $A \circ B$ is a sum of rank one non-negative def. matrices #

However, it is not true that $A, B \geq 0 \implies A \cdot B \geq 0$
(usual matrix product)

This is because:

$$A = A^*, B = B^* \not\Longrightarrow AB = (AB)^* \quad (\text{true only if } AB = BA)$$

Examples:

- $a_{jk} = e^{i(\phi_j - \phi_k)}$ $\phi_j \in \mathbb{R}$ $\implies A = vv^* \geq 0$, with $v_j = e^{i\phi_j}$
- $b_{jk} = \frac{1}{x_j + x_k} = \int_0^\infty e^{-t(x_j + x_k)} dt \implies B \geq 0$ (Riemann sum of non-neg. def. matrices)
- $(A \circ B)_{jk} = \frac{e^{i(\phi_j - \phi_k)}}{x_j + x_k} \implies (A \circ B) \geq 0$
- $A = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$, $B = \begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix} \implies A \cdot B = \begin{pmatrix} 2 & 1 \\ 2 & 1 \end{pmatrix} \neq (AB)^*$

$\implies A \cdot B \ngeq 0$ (in the "complex" sense)

Simultaneous diagonalization (and its applications)

Thm: if $A > 0$ and $B = B^*$ (both $n \times n$ matrices),

then $\exists C$ $n\times n$ invertible such that $CAC^* = I$, $CBC^* = D$ diagonal

Pf: we know that $\exists V$ $n\times n$ invertible such that $A = VV^*$, i.e. $V^{-1}A(V^{-1})^* = I$

since $V^{-1}B(V^{-1})^*$ is Hermitian, $\exists U$ $n \times n$ unitary such that

$UV^{-1}B(V^{-1})^*U^* = D$ diagonal. Let therefore $C = UV^{-1}$:

we have both $CBC^* = D$ and $CAC^* = UIU^* = I$ #

Thm: the map $A \mapsto \log\det A$ is concave on the set

of $n\times n$ positive definite matrices.

Pf: We need to check that for any $\alpha \in (0,1)$, $A, B > 0$:

$$\log\det(\alpha A + (1-\alpha)B) \geq \alpha \log\det A + (1-\alpha)\log\det B \quad (1)$$

By simultaneous diagonalization, $\exists C$ $n\times n$ invertible such

× that $A = CC^*$ and $B = CDC^*$. Thus, (1) is equivalent to:

$$\log\det(\alpha I + (1-\alpha)D) \geq \alpha \log\det I + (1-\alpha)\log\det D$$

i.e. $\sum_{j=1}^{n} \log(\alpha + (1-\alpha)d_j) \geq 0 + (1-\alpha)\sum_{j=1}^{n}\log d_j$

× which holds because of the concavity of the log itself. #

Corollary: $A \mapsto \log\det A^{-1}$ is convex on $\{A > 0\}$

NB: more generally, $A \mapsto \operatorname{Tr} f(A)$ is convex on $\{A > 0\}$

if $f: \mathbb{R}_+ \to \mathbb{R}$ is convex

Partial order on $\{A \geq 0\}$

Def: $A \geq B$ if $A - B \geq 0$

Note that $A \leq cI$ iff all e.v. of A are less than or equal to c

Thm: a) $A \geq B > 0 \implies \operatorname{Tr} A \geq \operatorname{Tr} B$

b) $A \geq B > 0 \implies \det A \geq \det B$ (& same is true for log det)

Pf: By simultaneous diagonalization, $\exists\, C$ $n \times n$ invertible s.t.

$A = CC^*$ and $B = CDC^*$, so

exercise!

× $A \geq B$ iff $C(I-D)C^* \geq 0$ $\boxed{\text{iff}}$ $(I-D) \geq 0$

iff all $d_j \leq 1$

a) $\operatorname{Tr} A = \operatorname{Tr} CC^* = \sum_{j,k=1}^{n} |c_{jk}|^2$

$\operatorname{Tr} B = \operatorname{Tr}(CDC^*) = \sum_{j,k=1}^{n} d_k |c_{jk}|^2 \leq \sum_{j,k=1}^{n} |c_{jk}|^2 = \operatorname{Tr} A$ ✓

b) (*) $BA^{-1} = CDC^*(CC^*)^{-1} = CDC^{-1}$

BA^{-1} and D have therefore the same e.v., namely $d_j \leq 1$ (and $d_j \geq 0$)

so $\det(BA^{-1}) \leq 1$ i.e. $\det B \leq \det A$ ✓

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Note that "reciprocally", if $A \geq 0$ and $\operatorname{Tr} A \leq c$,

then all e.v. of A are less than or equal to c (since their sum is), i.e. $A \leq cI$

[but note also that $\operatorname{Tr}(cI) = nc$, not c]

(*) or: $\det B = \det(CDC^*) = \det(C^*C) \underbrace{\det D}_{\leq 1} \leq \det(C^*C) = \det A$ ✓

Further matrix inequalities

Hadamard's inequality (version I)

If $A \geq 0$, then $\det A \leq \prod_{j=1}^{n} a_{jj}$ (and equality holds if A is diagonal)

Proof

- If $\det A = 0$, then there is nothing to prove.
- If $\det A \neq 0$, then A is invertible, so all $a_{jj} > 0$

(recall that for $A \geq 0$, $a_{jj} = 0 \Rightarrow a_{jk} = 0 \;\; \forall k \neq j$)

Let $D = \mathrm{diag}(a_{11}^{-1/2}, \ldots, a_{nn}^{-1/2})$: $\det A \leq \prod_{j=1}^{n} a_{jj}$ iff $\det(DAD) \leq 1$, so we may as well assume that all diagonal entries of A are equal to 1, in which case:

$$\det A = \prod_{j=1}^{n} \lambda_j \underset{\substack{\uparrow \\ \text{arithmetic-geometric} \\ \text{mean inequality}}}{\leq} \left(\frac{1}{n}\sum_{j=1}^{n} \lambda_j\right)^{1/n} = \left(\frac{1}{n} \operatorname{Tr} A\right)^{1/n} = 1^{1/n} = 1$$

(or concavity of the log) #

Note on the other hand that it is <u>not</u> true that

$$A \leq \mathrm{diag}(a_{11}, \ldots, a_{nn})$$

even in the case where all a_{jj} are equal.

Counter-ex: A = all ones matrix: $I - A \not\geq 0$

(but $\det A = 0 \leq 1 = \det I$)

Hadamard's inequality (version II)

For any $n\times n$ matrix B,

$$|\det B| \le \prod_{j=1}^{n} \sqrt{\sum_{k=1}^{n} |b_{jk}|^2} \quad \left(\begin{array}{l}\text{and equality holds if the}\\ \text{rows of } B \text{ are orthogonal}\end{array}\right)$$

Proof

Let $A = BB^*$: $|\det B| = \sqrt{\det A} \le \prod_{j=1}^{n} \sqrt{a_{jj}} = \prod_{j=1}^{n} \sqrt{\sum_{k=1}^{n} |b_{jk}|^2}$ #

Block Hadamard or Fischer's inequality

Let A be a $n\times n$ matrix, B be a $n\times m$ matrix, C be a $m\times m$ matrix such that $M = \begin{pmatrix} A & B \\ B^* & C \end{pmatrix}$ is positive definite

× (this is true iff $A > 0$ and $C - B^*A^{-1}B > 0$)

× Then $\det M \le \det A \cdot \det C$

Proof

Let $X = -A^{-1}B$. We have

× $$\begin{pmatrix} I & 0 \\ X^* & I \end{pmatrix}\begin{pmatrix} A & B \\ B^* & C \end{pmatrix}\begin{pmatrix} I & X \\ 0 & I \end{pmatrix} = \begin{pmatrix} A & 0 \\ 0 & C - B^*A^{-1}B \end{pmatrix}$$

so $\det\begin{pmatrix} A & B \\ B^* & C \end{pmatrix} = \underbrace{\det A \cdot \det(C - B^*A^{-1}B)}$

× Since $C - \underbrace{B^*A^{-1}B}_{\ge 0} \le C$, we have $\le \det A \cdot \det C$ #

This inequality generalizes in an obvious manner to the situation with more than 2 blocks.

Further inequalities (without proofs)

- Oppenheim's inequality:

$$A, B \geq 0 \implies \det(A \circ B) \geq (\det A) \cdot \prod_{j=1}^{n} b_{jj}$$

(with $B = I$, we recover Hadamard's inequality)

Corollary: $\det(A \circ B) \geq \det A \cdot \det B$

- By concavity of $A \mapsto \log \det A$, we have:

$$\begin{cases} A, B > 0 \\ \alpha \in (0,1) \end{cases} \implies (\det A)^{\alpha} (\det B)^{1-\alpha} \leq \det(\alpha A + (1-\alpha) B)$$

- Minkowski's inequality:

$$\begin{cases} A, B > 0 \\ n \times n \text{ matrices} \end{cases} \implies (\det(A+B))^{1/n} \geq (\det A)^{1/n} + (\det B)^{1/n}$$

and equality holds iff $B = cA$ for some $c > 0$ (scalar)

- Lieb's inequality:

$$A, B \geq 0 \implies \log \det(I + A + B) \leq \log \det(I + A) + \log \det(I + B)$$

- Finally, note this additional property:

for any $m \times n$ matrix B, the map

$A \mapsto \log \det (I + B A^{-1} B^{*})$ is convex

on the set of $n \times n$ positive definite matrices