Recall: \[ \mathbb{E}(m_{2e}^{(n)}) = \frac{1}{n^{2e}} \sum_{j_1 \cdots j_{2e}} \mathbb{E}(h_{j_1 j_2} h_{j_2 j_3} \cdots h_{j_{2e-1} j_{2e}}) \]

We are going to show that for any \( \varepsilon > 0 \),

\[ |\mathbb{E}(m_{2e}^{(n)}) - t_e| = O\left(\frac{1}{n^4}\right) \] (1)

where \( t_e \) are the Catalan numbers.

Notation: \( \mathcal{J}_{2e} = (j_1, \ldots, j_{2e}) \), \( h(\mathcal{J}_{2e}) = h_{j_1 j_2} h_{j_2 j_3} \cdots h_{j_{2e-1} j_{2e}} \)

\[ \mathbb{E}(m_{2e}^{(n)}) = \frac{1}{n^{2e}} \sum_{\mathcal{J}_{2e}} \mathbb{E}(h(\mathcal{J}_{2e})) \]

To each sequence \( \mathcal{J}_{2e} \), associate a directed graph \( g(\mathcal{J}_{2e}) \):

We say that two sequences \( \mathcal{J}_{2e} \) and \( \mathcal{J}'_{2e} \) are equivalent if their corresponding graphs are the same:

Notation: \( g_{2e} = g(\mathcal{J}_{2e}) = g(\mathcal{J}'_{2e}) \) or \( \mathcal{J}_{2e} \sim \mathcal{J}'_{2e} \)

\( \mathcal{J} \) with labelled edges:

\( (1, 2, 3, \ldots) \)
Because the $h_{2e}$ are identically distributed, and the corresponding graphs are the same,

$$E(h(\gamma_{2e})) = E(h(\gamma_{2e})) \quad \text{if} \quad \gamma_{2e} \sim \gamma_{2e}$$

So

$$E(m_{2e}^{(n)}) = \frac{1}{n^{1+e}} \sum_{\gamma_{2e}} \frac{E(h(\gamma_{2e}))}{\mathcal{Q}(\gamma_{2e})}$$

$$= \frac{1}{n^{1+e}} \sum_{\gamma_{2e}} \left( \frac{\# \{ \gamma_{2e} \in \gamma_{2e} \} \mathcal{Q}(\gamma_{2e})}{\mathcal{Q}(\gamma_{2e})} \right)$$

Let $V(\gamma_{2e})$ be the set of vertices of $\gamma_{2e}$ and $|V(\gamma_{2e})|$ be the number of such vertices. We have:

$$\# \{ \gamma_{2e} \in \gamma_{2e} \} = \text{number of possibilities of placing } |V(\gamma_{2e})| \text{ ordered points on } 1 \ldots n?$$

$$= n \cdot (n-1) \cdots (n-|V(\gamma_{2e})| + 1)$$

$$= n \cdot |V(\gamma_{2e})| \left( 1 + O\left( \frac{1}{n} \right) \right)$$

So

$$E(m_{2e}^{(n)}) = \frac{1}{n^{1+e}} \sum_{\gamma_{2e}} n \cdot |V(\gamma_{2e})| \mathcal{Q}(\gamma_{2e}) \left( 1 + O\left( \frac{1}{n} \right) \right)$$

Notice that if an edge appears an odd number of times in the graph, then for the same reason as last time:

$$\mathcal{Q}(\gamma_{2e}) = E(h(\gamma_{2e})) = E(h_{j_{1}j_{2}} \ldots h_{j_{2}j_{3}}) = 0$$

(\text{*) assumptions (i) \& (iii) \*)
Therefore, \( Q(g_{2e}) \geq \) if every edge in the graph appears an even number of times (the direction is indifferent, since \( h_{kij} = h_{jik} \)).

Terminology: the graph is said to be even in this case.

- In an even graph, each edge appears at least twice, so each "new" vertex costs at least two edges, therefore \( |V(g_{2e})| \leq 1 + e \) (since \# edges = \( 2e \))

\[
E(m_{2e}(n)) = \sum_{g_{2e} \text{ even}} \frac{1}{n+e} \cdot \frac{1}{|V(g_{2e})|} \cdot Q(g_{2e})(1 + O\left(\frac{1}{n}\right))
\]

- By assumption (ii), \( |Q(g_{2e})| \leq \frac{1}{e} E(h_{jije} \ldots h_{jjej})e \leq C^{2e} \)

independently of \( n \), so the only graphs contributing in a non-negligible manner to the above sum, are those for which \( |V(g_{2e})| = 1 + e \), i.e.

\[
E(m_{2e}(n)) = \sum_{g_{2e} \text{ even}} \frac{1}{n+e} \cdot \frac{1}{|V(g_{2e})|} \cdot Q(g_{2e})(1 + O\left(\frac{1}{n}\right))
\]
Finally, for an even graph \( g_{2e} \) such that \( |V(g_{2e})| = 1 + e \), each edge appears exactly twice.

Illustration:

\[
1 + k = 5, \; 2k = 8
\]

So \( \mathbb{R}(g_{2e}) = \mathbb{E}(h_{j:j+2}) \cdots \mathbb{E}(h_{j:j+2}) = 1 \) (by assumption (iv)).

and \( \mathbb{E}(W^{(n)}_{2e}) = \# \{ g_{2e} \text{ even } : |V(g_{2e})| = 1 + e \} + O\left(\frac{1}{n}\right) \)

How many even (and rooted) graphs are there with \( 2e \) branches and \( e+1 \) vertices on the line?

Illustration: \( e = 3 \)

Unfold:

\[
\text{i.e. } \# \{ g_{2e} \text{ even } : |V(g_{2e})| = 1 + e \}
\]

\[
= \# \{ \text{planar planted rooted trees with } e \text{ branches } \} = t_e
\]

So \( |\mathbb{E}(W^{(n)}_{2e}) - t_e| = O\left(\frac{1}{n}\right) \); this concludes the proof of (ii).
What about (2): \( \text{Var}(m^{(n)}_e) = O\left(\frac{1}{n^2}\right) \)?

First remark: this behaviour of the variance is atypical!

- Indeed, \( m^{(n)}_e = \frac{1}{n} \sum_{j=1}^{n} (A^{(n)j})^e \).

If the random variables \( x^{(n)}_j \) were iid, then we would have

\[
\text{Var}(m^{(n)}_e) = \frac{1}{n^2} \sum_{j=1}^{n} \text{Var}(A^{(n)j})^e = \frac{1}{n} \text{Var}(A^{(n)})^e = O\left(\frac{1}{n}\right)
\]

But the eigenvalues of a random matrix are everything but iid (as already seen from the joint distribution of the eigenvalues of the GOE at finite \( n \)), which explains the different behaviour of the variance.

- A simple heuristic for explaining the \( O\left(\frac{1}{n^2}\right) \) is the following: \( m^{(n)}_e = \frac{1}{n} \text{Tr}(A^{(n)})^e \);

\( m^{(n)}_e \) can therefore be seen as a function of the order \( n^2 \) iid entries of the matrix \( A^{(n)} \), which "explains" the variance of order \( \frac{1}{n^2} \), as opposed to the classical case with \( n \) iid random variables and variance \( \frac{1}{n} \).

(*) a rigorous proof of (2) can be found in Jonssoon 82
Let us shift our attention from \( m^{(n)}_e = \frac{1}{n} \sum_{j=1}^{n} f(x_j^{(n)})^2 \)

to \( \frac{1}{n} \sum_{j=0}^{n} f(x_j^{(n)}) \), where \( f: \mathbb{R} \to \mathbb{R} \) is a continuous

function satisfying moreover:

(i) \( f \) is convex

(ii) \( f \) is Lipschitz with constant \( L \), i.e. \( |f(x) - f(y)| \leq L| x - y | \)

Remember that \( \frac{1}{n} \sum_{j=1}^{n} f(x_j^{(n)}) = \frac{1}{n} \text{Tr} \left( f(H^{(n)}) \right) \)

So that this object can also be seen as a function of the order \( n^2 \) iid entries of the matrix \( H^{(n)} \).

Theorem (Ganemmel - Zeitouni 2006)

Under assumptions (i), (ii), (iii) - (vi),

\[
P \left( \left| \frac{1}{n} \sum_{j=1}^{n} f(x_j^{(n)}) - \int_{\mathbb{R}} f(x) \, \rho_{n}(x) \, dx \right| > t \right) \\ \leq 4 \exp \left( -n^2 \left( t - O(\frac{1}{n}) \right)^2 / 16 C^2 L^2 \right) \quad \forall t > 0
\]

More concretely, this theorem says that

\[
\frac{1}{n} \sum_{j=1}^{n} f(x_j^{(n)}) = \int_{\mathbb{R}} f(x) \, \rho_{n}(x) \, dx + O(\frac{1}{n})
\]

(in the probabilistic sense)

i.e. that the variance is of order \( \frac{1}{n^2} \) again.
Proof idea

\[ \frac{1}{n} \sum_{j=1}^{n} f(H^{(j)}) = \frac{1}{n} \text{Tr} \left( \frac{1}{n} H^{(j)} \right) = F_n(\{ \frac{1}{n} h_j, j \leq k \}) \in [-c, c] \]

It can be shown that

\[ f: \mathbb{R} \rightarrow \mathbb{R} \text{ convex} \implies F_n: [-c, c]^n \rightarrow \mathbb{R} \text{ is convex} \]

\[ f: \mathbb{R} \rightarrow \mathbb{R} \text{ Lipschitz with constant } L \implies F_n: [-c, c]^n \rightarrow \mathbb{R} \text{ is Lipschitz with } \frac{L}{n} \]

\[ \text{i.e. } |F_n(u) - F_n(v)| \leq \frac{L}{n} |u - v| \forall u, v \in [-c, c]^n \]

- **Talagrand's concentration inequality (Annals of Prob. 1996):**

\[
\begin{cases}
\text{If } Y_1, Y_n \text{ are iid random variables such that } |Y_i| \leq c \\
\text{and } F: [-c, c]^n \rightarrow \mathbb{R} \text{ is convex and Lipschitz with constant } K,
\end{cases}
\]

\[ \text{then } \mathbb{P}(|F(Y_1, \ldots, Y_n) - \tilde{F}_F| \geq t) \leq 4 \exp\left(-\frac{t^2}{16c^2K^2}\right) \]

where \( \tilde{F}_F \) is the median of \( F(Y_1, \ldots, Y_n) \).

- Here, \( n_0 = \frac{n(n-1)}{2} \) and \( K = \frac{L}{n} \), so

\[ \mathbb{P}(|F_n(\frac{1}{n} h_j, j \leq k)| - \tilde{F}_{F_n}| \geq t) \leq 4 \exp\left(-\frac{n^2K^2}{16c^2L^2}\right) \]

- The last step consists in showing that

\[ \tilde{F}_{F_n} = \mathbb{E}(F_n) + O\left(\frac{1}{n}\right) = \int_{\mathbb{R}} f(x) p_n(x) \, dx + O\left(\frac{1}{n}\right) \]