Random matrix theory: lecture 14

Proof of Wigner's theorem (cont'd)

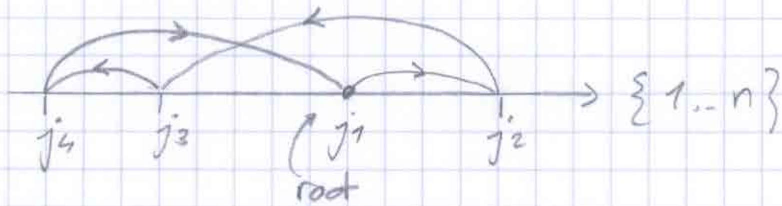
iid random variables ($\& \; h_{kj} = h_{jk}$)

Recall: $$\mathbb{E}\left(m_{2\ell}^{(n)}\right) = \frac{1}{n^{1+\ell}} \sum_{j_1 \cdots j_{2\ell}=1}^{n} \mathbb{E}\left(h_{j_1 j_2} h_{j_2 j_3} \cdots h_{j_{2\ell} j_1}\right)$$

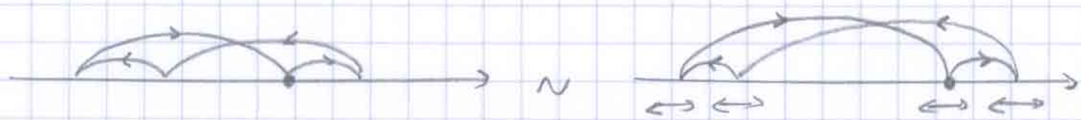
We are going to show that for any $\ell \geq 0$,
$$\left| \mathbb{E}\left(m_{2\ell}^{(n)}\right) - t_\ell \right| = O\left(\frac{1}{n}\right) \qquad (1)$$
where t_ℓ are the Catalan numbers.

- notation: $J_{2\ell} := (j_1 \cdots j_{2\ell})$, $h(J_{2\ell}) := h_{j_1 j_2} h_{j_2 j_3} \cdots h_{j_{2\ell} j_1}$
$$\Rightarrow \mathbb{E}\left(m_{2\ell}^{(n)}\right) = \frac{1}{n^{1+\ell}} \sum_{J_{2\ell}} \mathbb{E}\left(h(J_{2\ell})\right)$$

To each sequence $J_{2\ell}$, associate a directed graph(*) $g(J_{2\ell})$:



We say that two sequences $J_{2\ell}$ and $J_{2\ell}'$ are equivalent ($J_{2\ell} \sim J_{2\ell}'$) if their corresponding graphs are the same:



notation: $g_{2\ell} := g(J_{2\ell}) = g(J_{2\ell}')$ or $J_{2\ell} \in g_{2\ell}$

(the graph is an equivalence class for the sequences)

(*) with labelled edges!
(12, 23, 34, etc ...)

- Because the h_{jk} are identically distributed, and the corresponding graphs are the same,

$$\mathbb{E}(h(J_{2e})) = \mathbb{E}(h(J_{2e}')) \quad \text{if } J_{2e} \sim J_{2e}'$$

So

$$\mathbb{E}(m_{2e}^{(n)}) = \frac{1}{n^{1+e}} \sum_{g_{2e}} \sum_{J_{2e} \in g_{2e}} \underbrace{\mathbb{E}(h(J_{2e}))}_{:= Q(g_{2e})}$$

$$= \frac{1}{n^{1+e}} \sum_{g_{2e}} (\#\{J_{2e} \in g_{2e}\})\, Q(g_{2e})$$

- Let $V(g_{2e})$ be the set of vertices of g_{2e} and $|V(g_{2e})|$ be the number of such vertices.

We have:

$\#\{J_{2e} \in g_{2e}\}$ = number of possibilities of placing $|V(g_{2e})|$ ordered points on $\{1..n\}$

$$= n(n-1)\cdots(n-|V(g_{2e})|+1)$$

$$= n^{|V(g_{2e})|}\left(1+O\left(\tfrac{1}{n}\right)\right)$$

So

$$\mathbb{E}(m_{2e}^{(n)}) = \frac{1}{n^{1+e}} \sum_{g_{2e}} n^{|V(g_{2e})|} Q(g_{2e})\left(1+O\left(\tfrac{1}{n}\right)\right)$$

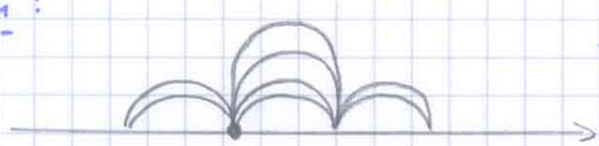
x - Notice that if an edge appears an odd number of times in the graph, then for the same reason as last time(*):

$$Q(g_{2e}) = \mathbb{E}(h(J_{2e})) = \mathbb{E}(h_{j_1 j_2} \cdots h_{j_{2e} j_1}) = 0$$

(*) assumptions (i) & (iii)

Therefore, $Q(g_{2\ell}) > 0$ iff every edge in the graph appears an even number of times (the direction is indifferent, since $h_{kj} = h_{jk}$).

Illustration:



even graph

Terminology: the graph is said to be even in this case

- In an even graph, each edge appears at least twice, so each "new" vertex costs at least two edges, therefore $|V(g_{2\ell})| \leq 1+\ell$ (since # edges $= 2\ell$)

and

$$\mathbb{E}(m_{2\ell}^{(n)}) = \frac{1}{n^{1+\ell}} \sum_{\substack{g_{2\ell} \text{ even} \\ |V(g_{2\ell})| \leq 1+\ell}} n^{|V(g_{2\ell})|} Q(g_{2\ell})\left(1+O\left(\tfrac{1}{n}\right)\right)$$

- By assumption (ii)', $|Q(g_{2\ell})| \leq |\mathbb{E}(h_{j_1 j_2} \cdots h_{j_{2\ell} j_1})| \leq C^{2\ell}$ independently of n, so the only graphs contributing in a non-negligible manner to the above sum are those for which $|V(g_{2\ell})| = 1+\ell$, i.e.

$$\mathbb{E}(m_{2\ell}^{(n)}) = \frac{1}{\cancel{n^{1+\ell}}} \sum_{\substack{g_{2\ell} \text{ even} \\ |V(g_{2\ell})| = 1+\ell}} \cancel{n^{1+\ell}}\, Q(g_{2\ell})\left(1+O\left(\tfrac{1}{n}\right)\right)$$

- Finally, for an even graph $g_{2\ell}$ such that $|V(g_{2\ell})| = 1+\ell$, each edge appears exactly twice.

Illustration:

$(1+k=5,\ 2k=8)$

So $Q(g_{2\ell}) = \mathbb{E}(h_{j_1j_2}^2) \cdots \mathbb{E}(h_{j_kj_1}^2) = 1$ (by assumption (iv)).

and $\mathbb{E}(m_{2\ell}^{(n)}) = \#\{g_{2\ell} \text{ even} : |V(g_{2\ell})| = 1+\ell\} + O(\frac{1}{n})$

- How many even (and rooted) graphs are there with 2ℓ branches and $\ell+1$ vertices on the line?

Illustration: $\ell = 3$

unfold:

i.e. $\#\{g_{2\ell} \text{ even} : |V(g_{2\ell})| = 1+\ell\}$

$= \#\{\text{planar planted rooted trees with } \ell \text{ branches}\} = t_\ell$

So $|\mathbb{E}(m_{2\ell}^{(n)}) - t_\ell| = O(\frac{1}{n})$; this concludes the proof of (1). #

What about (2)(*): $Var(m_\ell^{(n)}) = O(\frac{1}{n^2})$?

First remark: this behaviour of the variance is atypical!

- Indeed, $m_\ell^{(n)} = \frac{1}{n}\sum_{j=1}^{n} (\lambda_j^{(n)})^\ell$.

  If the random variables $\lambda_j^{(n)}$ were iid,

  then we would have

  $$Var(m_\ell^{(n)}) = \frac{1}{n^2}\sum_{j=1}^{n} Var((\lambda_j^{(n)})^\ell) = \frac{1}{n} Var((\lambda_1^{(n)})^\ell) = O(\tfrac{1}{n})$$

  But the eigenvalues of a random matrix are everything but iid (as already seen from the joint distribution of the eigenvalues of the GOE at finite n), which explains the different behaviour of the variance.

- A simple heuristics for explaining the $O(\frac{1}{n^2})$ is the following: $m_\ell^{(n)} = \frac{1}{n} Tr((H^{(n)})^\ell)$;

  $m_\ell^{(n)}$ can therefore be seen as a function of the order n^2 iid entries of the matrix $H^{(n)}$, which "explains" the variance of order $\frac{1}{n^2}$, as opposed to the classical case with n iid random variables and variance $\frac{1}{n}$.

(*) a rigorous proof of (2) can be found in $\begin{cases}\text{Jonsson 82}\\ \text{Anderson-Zeitouni 04}\end{cases}$

Concentration

Let us shift our attention from $m_\ell^{(n)} = \frac{1}{n}\sum_{j=1}^{n} (\lambda_j^{(n)})^\ell$ to $\frac{1}{n}\sum_{j=1}^{n} f(\lambda_j^{(n)})$, where $f: \mathbb{R} \to \mathbb{R}$ is a continuous function satisfying moreover:

(v) f is convex

(vi) f is Lipschitz with constant L, i.e. $|f(x)-f(y)| \leq L|x-y|$ $\forall x,y \in \mathbb{R}$

Remember that $\frac{1}{n}\sum_{j=1}^{n} f(\lambda_j^{(n)}) = \frac{1}{n}\mathrm{Tr}\,(f(H^{(n)}))$,

so that this object can also be seen as a function of the order n^2 iid entries of the matrix $H^{(n)}$.

Theorem (Guionnet - Zeitouni 2000)

Under assumptions (i), (ii)', (iii) — (vi),

$$\mathbb{P}\left(\left|\frac{1}{n}\sum_{j=1}^{n} f(\lambda_j^{(n)}) - \int_{\mathbb{R}} f(x)\, p_{rr}(x)\, dx\right| > t\right)$$

$$\leq 4 \exp\left(-n^2\left(t - O(\tfrac{1}{n})\right)^2 / 16\, c^2 L^2\right) \qquad \forall t > 0$$

recall that $|h_{jk}| \leq c$ (assumption (ii)')

× More concretely, this theorem says that

$$\frac{1}{n}\sum_{j=1}^{n} f(\lambda_j^{(n)}) = \int_{\mathbb{R}} f(x)\, p_{rr}(x)\, dx + O\left(\frac{1}{n}\right)$$

(in the probabilistic sense)

i.e. that the variance is of order $\frac{1}{n^2}$ again.

Proof idea

- $\frac{1}{n}\sum_{j=1}^{n} f(\lambda_j^{(n)}) = \frac{1}{n}\mathrm{Tr}\left(f(H^{(n)})\right) := F_n(\{\underbrace{h_{jk}}_{\in[-c,c]}, j\le k\})$

  It can be shown that

  - $f:\mathbb{R}\to\mathbb{R}$ convex $\Rightarrow F_n : [-c,c]^{\frac{n(n+1)}{2}} \to \mathbb{R}$ is convex
  - $f:\mathbb{R}\to\mathbb{R}$ Lipschitz with constant L

    $\Rightarrow F_n : [-c,c]^{\frac{n(n+1)}{2}} \to \mathbb{R}$ is Lipschitz with cst $\frac{L}{n}$

    i.e. $|F_n(u) - F_n(v)| \le \frac{L}{n}\|u-v\| \quad \forall u,v\in[-c,c]^{\frac{n(n+1)}{2}}$

- Talagrand's concentration inequality (Annals of Prob. 1995):

  If $Y_1 .. Y_m$ are iid random variables such that $|Y_j|\le c$
  and $F:[-c,c]^m \to \mathbb{R}$ is convex and Lipschitz with cst K,
  then $\mathbb{P}(|F(Y_1..Y_m) - M_F| \ge t) \le 4\exp\left(-\frac{t^2}{16c^2K^2}\right)$
  where M_F is the median of $F(Y_1 .. Y_m)$.

- Here, $m = \frac{n(n+1)}{2}$ and $K = \frac{L}{n}$, so

$$\mathbb{P}\left(|F_n(\{h_{jk}, j\le k\}) - M_{F_n}| \ge t\right) \le 4\exp\left(-\frac{n^2t^2}{16c^2L^2}\right)$$

- The last step consists in showing that

$$M_{F_n} = \mathbb{E}(F_n) + O\left(\tfrac{1}{n}\right) = \int_{\mathbb{R}} f(x)\, p_n(x)\,dx + O\left(\tfrac{1}{n}\right)$$

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