Random matrix theory: lecture 11

Distributions without random variables!

Def: a (probability) distribution on $\mathbb{R}$ is an application $\mu: \mathcal{B}(\mathbb{R}) \to [0, 1]$ such that:

\[
\begin{cases}
\mu(\emptyset) = 0, \quad \mu(\mathbb{R}) = 1 \\
\text{if } B_n \cap B_m = \emptyset \text{ for all } n \neq m, \text{ then } \mu\left(\bigcup_{n=1}^{\infty} B_n\right) = \sum_{n=1}^{\infty} \mu(B_n)
\end{cases}
\]

Remark: a distribution might therefore exist independently of any underlying random variable!

Def: the cumulative distribution function (cdf) associated to a distribution $\mu$ is the application $F_\mu: \mathbb{R} \to [0, 1]$ defined as $F_\mu(t) := \mu([\infty, t])$, $t \in \mathbb{R}$.

Properties:

\[
\begin{align*}
\lim_{t \to \infty} F_\mu(t) &= \mu(\mathbb{R}) = 1, \quad \lim_{t \to -\infty} F_\mu(t) = \mu(\emptyset) = 0 \\
F_\mu &\text{ is non-decreasing: } t_1 \leq t_2 \Rightarrow F_\mu(t_1) \leq F_\mu(t_2) \\
F_\mu &\text{ is right-continuous: } \lim_{\varepsilon \to 0^+} F_\mu(t + \varepsilon) = F_\mu(t) \quad \forall t \in \mathbb{R}
\end{align*}
\]

\[\text{x}\]

The knowledge of $F_\mu$ characterizes $\mu$ entirely!

(and reciprocally, of course)
Two canonical classes of distributions

A) discrete distributions:

\[ \exists D = \{ x_n \}_{n=1}^{\infty} \text{ (countable subset) such that } \mu(D) = 1 \]
(i.e., all the weight of the distribution \( \mu \) is on \( D \))

Let \( p_n = \mu(\{x_n\}) : \mu(B) = \sum_{x_n \in B} p_n, B \in B(\mathbb{R}) \)

and \( F_n(t) = \sum_{x_n \leq t} p_n \) step function.

B) continuous distributions:

\[ \mu(B) = 0 \text{ if } \{ \text{length of } B \} = 0 \]
(in particular: \( \mu(\{x\}) = 0 \) \( \forall x \in \mathbb{R} \))

\[ \Rightarrow \] There exists a function \( p_n \) such that \( p_n(x) \geq 0 \),
\[ \int_{\mathbb{R}} p_n(x) \, dx = 1 \text{ and } \mu(B) = \int_{B} p_n(x) \, dx, B \in B(\mathbb{R}) \]

\( p_n \) is the probability density function (pdf) of \( \mu \).

Moreover, \( F_n(t) = \int_{-\infty}^{t} p_n(x) \, dx \) smooth function.

\[ \text{NB: } p_n'(x) = F_n''(x) = \frac{dF_n}{dx}(x) \]
Riemann-Stieltjes Integral with respect to a distribution \( \mu \)

**Def:** A function \( f : \mathbb{R} \to \mathbb{R} \) is continuous if \( \forall a \in \mathbb{R} \) such that \( f(x) = 0 \) \( \forall |x| > A \) (fixed) and \( \mu \) be a (general) distribution on \( \mathbb{R} \). Let \( -A = a_0 < a_1 < \ldots < a_n = A \) be a subdivision of \( [-A, A] \).

Let \( I_n = \sum_{j=1}^{n} f(b_j) \mu([a_{j-1}, a_j]) \)

where \( b_j \) is any point in \( [a_{j-1}, a_j] \).

**Thm:**

For any continuous function \( f \) vanishing outside \( [-A, A] \) and any sequence of subdivisions such that \( \max |a_{j-1} - a_j| \to 0 \) \( n \to \infty \)

the sequence \( I_n \) converges to \( I = \int_{\mathbb{R}} f(x) \, d\mu(x) \) as \( n \to \infty \).

**Alternate notation:** \( I = \int_{\mathbb{R}} f(x) \, d\mu(dx) \)

since \( \mu([a_{j-1}, a_j]) = \mu(a_j) - \mu(a_{j-1}) \), one still writes

\[
I = \int_{\mathbb{R}} f(x) \, d\left( \frac{d\mu}{dx} \right) \quad \text{or} \quad I = \int_{\mathbb{R}} f(x) \, \mu'(dx)
\]

**Remark:**

The Riemann-Stieltjes integral can be extended to non-vanishing continuous functions on \( \mathbb{R} \) (letting \( A \to \infty \)).
Lebesgue's integral with respect to a distribution \( \mu \)

**Def:** a function \( f: \mathbb{R} \to \mathbb{R} \) is Borel-measurable if \( \forall \text{ all } a < f(x) < b \) is a Borel subset of \( \mathbb{R} \)

**NB:** This is a much weaker condition than being continuous!

- Let \( f \) be a non-negative Borel-measurable function on \( \mathbb{R} \)

and \( j_n := \sum_{j=1}^{\infty} \frac{j-1}{2^n} \cdot \mu \left( \left\{ x \in \mathbb{R} : \frac{j-1}{2^n} \leq f(x) < \frac{j}{2^n} \right\} \right) \)

**Remarks:** for fixed \( n \), \( j_n \in [0,\infty] \)

- \( j_n \leq j_{n+1} \quad \forall n \) (since the height of the steps is divided by 2 from \( n \) to \( n+1 \))

So \( \lim_{n \to \infty} j_n = J \) exists and belongs to \( [0,\infty] \).

\( J \) is the Lebesgue integral of \( f \) with respect to \( \mu \)

and is denoted as \( \int_{\mathbb{R}} f(x) \, d\mu(x) \). \( [ \text{ same notation as Riemann-Stieltjes integral} ] \)

- Let \( f \) be a Borel-measurable function on \( \mathbb{R} \) such that \( \int_{\mathbb{R}} |f(x)| \, d\mu(x) < \infty \). Then

\[ \int_{\mathbb{R}} f(x) \, d\mu(x) := \int_{\mathbb{R}} f^+(x) \, d\mu(x) - \int_{\mathbb{R}} f^-(x) \, d\mu(x) \]

where \( f^+(x) := \max(0, f(x)) \geq 0 \) and \( f^-(x) := \max(0, -f(x)) \geq 0 \).

\( (*) \) and \( \mu \) be a (general) distribution on \( \mathbb{R} \)
Remarks:

- Both Riemann–Stieltjes and Lebesgue's integrals are well defined for $f: \mathbb{R} \rightarrow \mathbb{R}$ bounded and continuous (and they coincide for such $f$). Let us check this for Lebesgue: $\int_{\mathbb{R}} |f(x)| \, d\mu(x) \leq \sup_{x \in \mathbb{R}} |f(x)| \cdot \int_{\mathbb{R}} d\mu(x) < \infty$.

- Lebesgue's integral is more general (except for some particular cases), so we will always refer implicitly to this second definition.

Special cases of distributions $\mu$:

A) Integral with respect to a discrete distribution $\mu$:

$$\int_{\mathbb{R}} f(x) \, d\mu(x) = \sum_{n=1}^{\infty} f(x_n) \, \mu_n$$

where $\mu_n = \mu(\{x_n\})$

B) Integral with respect to a continuous distribution $\mu$:

$$\int_{\mathbb{R}} f(x) \, d\mu(x) = \int_{\mathbb{R}} f(x) \, \mu'(x) \, dx$$

where $\mu'(x) = \frac{d\mu}{dx}$
Special cases of functions $f$:

1) For a given $t \in \mathbb{R}$, let $f(x) = \mathbf{1}_{]a, b[}(x)$. $f$ is Borel-measurable and bounded and
   $$\int_{\mathbb{R}} \mathbf{1}_{]a, b[}(x) \, d\mu(x) = \mu(]a, b[) = F_{\mu}(t),$$
   CDF of $\mu$.

2) For a given $t \in \mathbb{R}$, let $f(x) = e^{itx}$; $f$ is bounded and continuous and
   $$\int_{\mathbb{R}} e^{itx} \, d\mu(x) = \phi_{\mu}(t),$$
   Fourier transform or characteristic function of $\mu$.

3) For a given $k > 0$, let $f(x) = x^k$; $f$ is continuous and unbounded; $\int_{\mathbb{R}} |f(x)| \, d\mu(x)$ is therefore not necessarily finite. When this is the case, we define
   $$\int_{\mathbb{R}} x^k \, d\mu(x) = M_k, \text{ moment of order } k \text{ of } \mu.$$

4) For a given $z \in \mathbb{C} \setminus \mathbb{R}$, let $f(x) = \frac{1}{x - z}$; $f$ is complex-valued, bounded and continuous (since $z \notin \mathbb{R}$)
   $$\int_{\mathbb{R}} \frac{1}{x - z} \, d\mu(x) = g_{\mu}(z),$$
   Stieltjes transform of $\mu$. 

$\times$
Weak convergence of sequences of distributions

**Def:** A sequence of distributions \((F_n)_{n=1}^{\infty}\) converges weakly to a distribution \(F\) if

\[\lim_{n \to \infty} F_{n}(t) = F(t)\]

\(\forall t\in\mathbb{R}\) continuity point of \(F\)

(Unfortunate) notation: \(F_n \Rightarrow F\)

*Two equivalent definitions:*

**Def':** \(F_n \Rightarrow F\) iff \(\lim_{n \to \infty} F_n([a,b]) = F([a,b])\)

\(\forall a < b\) such that \(F([a,b]) = F([a,b]) = 0\)

*(Proof: use \(F([a,b]) = F([b]) - F([a])\))*

**Def":** \(F_n \Rightarrow F\) iff \(\lim_{n \to \infty} \int_{\mathbb{R}} f(x) \, dF_n(x) = \int_{\mathbb{R}} f(x) \, dF(x)\)

\(\forall f: \mathbb{R} \to \mathbb{R}\) bounded and continuous

*(Proof: approximate \(\int_{[a,b]} \) by a sequence of bold and continuous functions; reciprocally, approximate \(f\) by a sequence of step functions)*
Weak convergence and Fourier transform

Proposition (inversion formula)

The knowledge of the function \( \phi_n(t) = \int \frac{e^{itx}}{2\pi} dx \), \( t \in \mathbb{R} \)
characterizes \( \mu \) entirely. Moreover, \( \forall a < b, \)

\[
\mu([a, b]) = \frac{1}{\sqrt{2\pi}} \int_{-a}^{b} e^{-itx} \phi_n(t) \, dt.
\]

If \( \mu \) is a continuous distribution with pdf \( p_n \), then

\[
p_n(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-itx} \phi_n(t) \, dt.
\]

Proposition:

\[
\mu_n \Rightarrow \mu \quad \text{iff} \quad \lim_{n \to \infty} \phi_n(t) = \phi(t) \quad \forall t \in \mathbb{R}
\]

Remarks:

- This proposition is of utmost importance in probability
  (it is used for proving the central limit theorem, e.g.)

- Unfortunately, it is mostly useless for random matrices
  (explanation coming)