Exercises November 27, 2009. Quantum information theory and computation


We consider a source of mixed states $\rho_x$ occurring each with probabilities $p_x$. Messages are $N$ letter strings of the form $\rho_{x_1} \otimes \ldots \otimes \rho_{x_N}$ and have a probability $p_{x_1}\ldots p_{x_N}$. In this exercise we want to give some support to the conjecture that the achievable rate of compression for a source of mixed states is equal to the Holevo quantity

$$\chi(\{p_x, \rho_x\}) = S(\rho) - \sum_x p_x S(\rho_x), \quad \rho = \sum_x p_x \rho_x$$

Note that in the case of a source of pure states $\rho_x = |\phi_x\rangle\langle\phi_x|$ the Holevo quantity $\chi(\rho)$ reduce to $S(\rho)$ which is the optimal achievable rate given by Schumacher’s theorem.

a) Take a source constituted of the unique letter $\rho_0$ occurring with probability $p_0 = 1$. How many bits are needed to compress this source? What is the value of $\chi(\rho)$? Is this consistent?

b) Now consider a source of mixed mutually orthogonal states. Two mixed states are said to be mutually orthogonal if

$$\text{Tr} \rho_x \rho_y = 0, \quad x \neq y$$

Construct purifications $|\Psi_x\rangle$ of $\rho_x$ that satisfy (hint: use the spectral decomposition)

$$\langle \Psi_x | \Psi_y \rangle = 0, \quad x \neq y$$

What would be an encoding scheme achieving a compression rate of $H(X) = -\sum_x p_x \log p_x$? Why would this rate be optimal? Check that in the present case we have

$$H(X) = \chi(\rho)$$

hint: no big calculations.

Exercise 2. Product state capacity of the quantum depolarizing channel.

We want to calculate the capacity of the depolarizing channel with noise strength $0 \leq \epsilon \leq 1$. This channel is defined by the map (take the opportunity to check the formulas of the course, although you don’t really need them for this exercise)

$$Q(\rho) = (1 - \epsilon)\rho + \frac{\epsilon}{2}I$$
The product state capacity is given by

\[ C(\epsilon) = \max_{p_x, \rho_x} \chi\left(\{p_x, Q(\rho_x)\}\right) \]

where the supremum is taken over a finite alphabet \( \{\rho_x\} \) of \( 2 \times 2 \) density matrices and a probability distribution over this alphabet. We will accept (and there is a proof) that the max is attained for pure states \( \rho_x = |\phi_x\rangle \langle \phi_x| \) (not necessarily orthogonal).

Prove that

\[ C(\epsilon) = \ln 2 - H\left(\frac{\epsilon}{2}\right) \]

were \( H(x) \) is the usual binary entropy function (defined with natural log).

*hint: no big optimization.*