Exercises september 25, 2009. Quantum Information Theory and Computation.

Problem 1. Polarisation measurements and uncertainty relation.

If we pass photons through a polariser at an angle θ they are prepared in the state $|\theta\rangle = \cos\theta|x\rangle + \sin\theta|y\rangle$. We are going to analyse them with two different analysers: one at an angle α and the other at an angle β. A detector records the photons just after the analyser. For the analyser α we record the number $p_\alpha = \pm 1$ according to the fact that a photon is detected or not detected. For the analyser β we record the number $p_\beta = \pm 1$ according to the fact that a photon is detected or not detected.

a) Compute the probabilities of detection and non detection, $Prob(p_\alpha = \pm 1)$ and $Prob(p_\beta = \pm 1)$ for both analysers.

b) Compute the expectation value and variance of the random variables p_α, p_β.

c) Consider now the "observables" $P_\alpha = (+1)|\alpha\rangle\langle\alpha| + (-1)|\alpha_\perp\rangle\langle\alpha_\perp|$ and $P_\beta = (+1)|\beta\rangle\langle\beta| + (-1)|\beta_\perp\rangle\langle\beta_\perp|$. Check that the expectation and variance of $p_{\alpha,\beta}$ are equal to (here $\phi = \alpha, \beta$)

$$Exp[p_\phi] = \langle\theta|P_\phi|\theta\rangle, \qquad Var[p_\phi] = \langle\theta|P_\phi^2|\theta\rangle - \langle\theta|P_\phi|\theta\rangle^2$$

d) Compute the commutator $[P_\alpha, P_\beta] = P_\alpha P_\beta - P_\beta P_\alpha$. Check that Heisenberg's uncertainty principle is satisfied for any $|\theta\rangle$.

$$\Delta P_\alpha \Delta P_\beta \geq \frac{1}{2}|\langle\theta|[P_\alpha, P_\beta]|\theta\rangle|$$

Here $\Delta P = (Var[p])^{1/2}$.

Remark: you can write the matrices corresponding to P_α and P_β in the computational basis to see how they look like. But the above calculations are more easily done directly in Dirac notation instead of matrix form.