Homework Set #9

Due 18 December 2009, 6 pm, in INR036

Problem 1 (Impulse power)

Consider the additive white Gaussian channel $Y_i = X_i + Z_i$ (Fig 1), where $Z_i \sim N(0, N)$, and the input signal has average power constraint P.

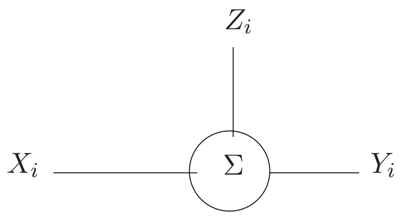

Figure 1: channel of problem 1

(a) Suppose that we use all our power at time 1 (i.e., $EX_1^2 = np$ and $EX_i^2 = 0$ for $i = 2, 3, \cdots, n$). Find

$$\max_{f(x^n)} \frac{I(X^n; Y^n)}{n},$$

where the maximization is over all distributions $f(x^n)$ subject to the constraint $EX_1^2 = np$ and $EX_i^2 = 0$ for $i = 2, 3, \cdots, n$.

(b) Find

$$\max_{f(x^n): E(\frac{1}{n}\sum_{i=1}^n X_i^2) \le P} \frac{1}{n} I(X^n, Y^n),$$

and compare to part (a).

Problem 2 (differential conditional entropy)

Consider the noisy channel $\mathbf{Y} = \mathbf{X} + \mathbf{Z}$ where $\mathbf{X}$, $\mathbf{Y}$ and $\mathbf{Z}$ are $n \times 1$ real vectors and $\mathbf{Z} \sim \mathcal{N}(0, \mathbf{I})$. As depicted in Fig 2, in order to estimate $\mathbf{X}$ from the received signal $\mathbf{Y}$, a linear detector $\mathbf{F}$ follows $\mathbf{Y}$ so that $\hat{\mathbf{X}} = \mathbf{F}\mathbf{Y}$.

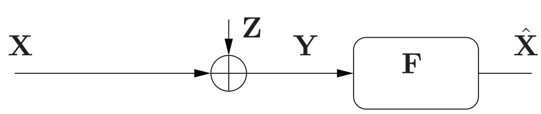

Figure 2: Noisy channel of problem 2

(a) Find $\mathbf{F}$ so that the mean square error is minimized, i.e., $\mathbb{E}(\|\hat{\mathbf{X}} - \mathbf{X}\|^2)$ is minimized. Hint: You should prove the orthogonality principle: mean square error is minimized if $\mathbb{E}((\hat{\mathbf{X}} - \mathbf{X})\mathbf{Y}^t) = 0$.

(b) Find $\mathbb{E}(\|\hat{\mathbf{X}} - \mathbf{X}\|^2)$ for the $\mathbf{F}$ you found in part (a).

(c) Find $h(\hat{\mathbf{X}}|\mathbf{X})$.

(d) Relate parts (b) and (c).

Problem 3 (Side-information channel)

Consider a Gaussian channel as shown in Fig. 3 with the input power constraint $\mathbb{E}[X_1^2] \leq P$. The receiver observes the noisy signal

$$Y_1 = X_1 + Z_1,$$

where Z_1 is Gaussian noise distributed as $Z_1 \sim \mathcal{N}(0, N)$ and independent of the channel inputs. The receiver has also access to a Gaussian random variable $U_1 \sim \mathcal{N}(0, N)$ which is independent of X_1, but correlated with Z_1, *i.e.,* $\mathbb{E}[Z_1 U_1] = \mu_1 N$.

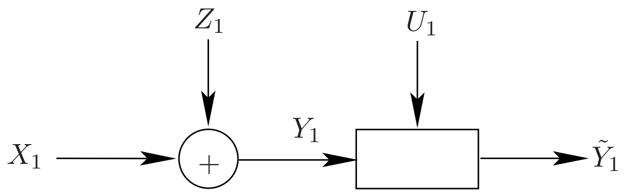

Figure 3: A Gaussian channel with side information at the receiver

(a) It can be shown that the capacity of the described channel is $C = \max_{p(x)} I(Y_1, U_1; X_1)$. Calculate C.
Hint: Start by writing $I(Y_1, U_1; X_1) = h(Y_1, U_1) - h(Y_1, U_1|X_1) = h(Y_1, U_1) - h(Z_1, U_1)$.

(b) Let us apply a function on Y_1 and U_1 at the receiver to obtain $\tilde{Y}_1 = Y_1 + \gamma U_1$. Show that $I(\tilde{Y}_1; X_1) \leq I(Y_1, U_1; X_1)$. Assuming $X_1 \sim \mathcal{N}(0, P)$, find the value of γ, for which $\tilde{Y}_1$ would be a sufficient statistic for decoding X_1, *i.e.,* $I(\tilde{Y}_1; X_1) = I(Y_1, U_1; X_1)$.

(c) Define $\hat{X}_1 = \alpha_1 Y_1 + \beta_1 U_1$. Find the optimal values for α_1 and β_1 to minimize $\mathbb{E}[(\hat{X}_1 - X_1)^2]$.
Hint:

$$\begin{aligned}
\mathbb{E}[(\hat{X}_1 - X_1)^2] &= \mathbb{E}[(\alpha_1 Y_1 + \beta_1 U_1 - X_1)^2] \\
&= \mathbb{E}[((\alpha_1 - 1)X_1 + \alpha_1 Z_1 + \beta_1 U_1)^2] \\
&= \mathbb{E}[((\alpha_1 - 1)X_1)^2 + (\alpha_1 Z_1 + \beta_1 U_1)^2] \\
&= (\alpha_1 - 1)^2 \mathbb{E}[X_1^2] + \alpha_1^2 \mathbb{E}[Z_1^2] + \beta_1^2 \mathbb{E}[U_1^2] + 2\alpha_1\beta_1 \mathbb{E}[Z_1 U_1]
\end{aligned}$$

Now assume that we have two parallel channels where each channel looks like the channel we considered above, as shown in Fig 4. The correlation between the Gaussian noises is of the form

$$\mathbb{E}\left[\begin{bmatrix} Z_1 \\ U_1 \\ Z_2 \\ U_2 \end{bmatrix}\begin{bmatrix} Z_1 & U_1 & Z_2 & U_2 \end{bmatrix}\right] = \begin{bmatrix} N & \mu_1 N & 0 & 0 \\ \mu_1 N & N & 0 & 0 \\ 0 & 0 & N & \mu_2 N \\ 0 & 0 & \mu_2 N & N \end{bmatrix}.$$

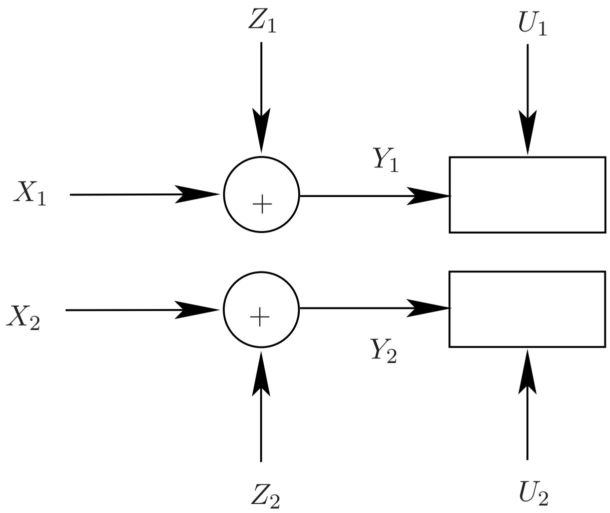

Figure 4: Two parallel Gaussian channels with side information at the receivers

It can be shown that

$$\max_{p(x_1,x_2):\mathbb{E}[X_1^2]+\mathbb{E}[X_2^2]\le P} I(X_1,X_2;U_1,Y_1,U_2,Y_2) = \max_{p(x_1,x_2)=p(x_1)p(x_2):\mathbb{E}[X_1^2]+\mathbb{E}[X_2^2]\le P} \{I(X_1;U_1,Y_1)+I(X_2;U_2,Y_2)\};$$

i.e., it is optimal to choose independent X_1 and X_2.

(d) Find the capacity of this parallel channel with a total power constraint $\mathbb{E}[X_1^2]+\mathbb{E}[X_2^2] \le P$. What is the optimal power allocation for this channel, *i.e.*, what are the optimal values of P_1 and P_2 such that $\mathbb{E}[X_1^2] = P_1$ and $\mathbb{E}[X_2^2] = P_2$ and $P_1 + P_2 \le P$.

Hint: Use the result of part **(a)**.

Problem 4 (**Multipath Gaussian channel**)

Consider a Gaussian noise channel with power constraint P, where the signal takes two different paths and the received noisy signals are added together at the antenna, see Fig 5.

(a) Find the capacity of this channel if Z_1 and Z_2 are jointly normal with covariance matrix

$$K_Z = \begin{bmatrix} \sigma^2 & \rho\sigma^2 \\ \rho\sigma^2 & \sigma^2 \end{bmatrix}.$$

(b) What is the capacity for $\rho = 0$, $\rho = 1$, $\rho = 1$?

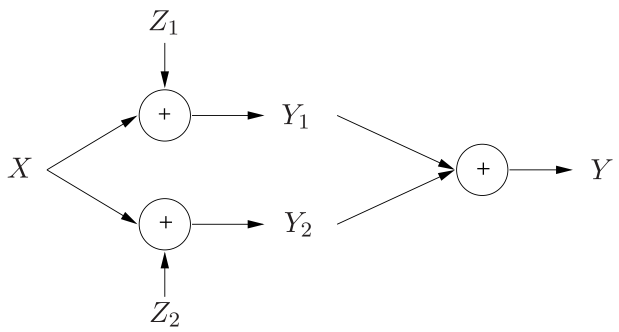

Figure 5: Multipath Gaussian channel.