Homework Set #8

Due 11 December 2009, 6 pm, in INR036

Problem 1 (Channel Capacity)

Consider a discrete memoryless channel (DMC) with 3 inputs and 3 outputs. Let $\mathcal{X} = \{1,2,3\}$ and $\mathcal{Y} = \{1,2,3\}$ be the input and output alphabets and the transition probabilities of the channel are given by $P_{Y|X}(y=i|x=j) = M_{i,j}$ where

$$\mathbf{M} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1-\epsilon & \epsilon \\ 0 & \epsilon & 1-\epsilon \end{bmatrix}.$$

(a) Find the capacity of the channel and the optimal input distribution in terms of ϵ.
Hint: Try to guess the (form of) optimal input distribution and check the Kuhn-Tucker conditions for your guess.

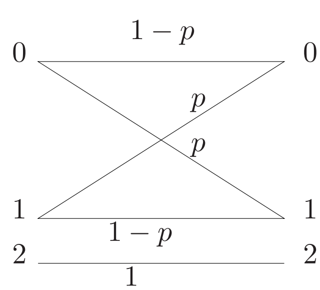
(b) Explain what happens when $\epsilon = 0$ and $\epsilon = \frac{1}{2}$.

Problem 2 (Parallel Channels and Choice of channels)

(a) Find the capacity C of the union of two channels $(\mathcal{X}_1, p_1(y_1|x_1), \mathcal{Y}_1)$ and $(\mathcal{X}_2, p_2(y_2|x_2), \mathcal{Y}_2)$, where at each time, one can send a symbol over channel 1 or channel 2 but not both. Assume that the output alphabets are distinct and do not intersect. Show that $2^C = 2^{C_1} + 2^{C_2}$. Thus 2^C is the effective alphabet size of a channel with capacity C.

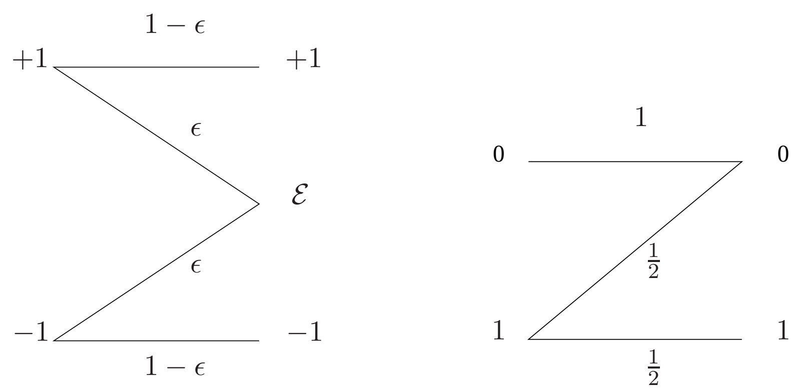
(b) Consider two discrete memoryless channels $(\mathcal{X}_1, p(y_1|x_1), \mathcal{Y}_1)$ and $(\mathcal{X}_2, p(y_2|x_2), \mathcal{Y}_2)$ with capacities C_1 and C_2 respectively. A new channel $(\mathcal{X}_1 \times \mathcal{X}_2, p(y_1|x_1) \times p(y_2|x_2), \mathcal{Y}_1 \times \mathcal{Y}_2)$ is formed in which $x_1 \in \mathcal{X}_1$ and $x_2 \in \mathcal{X}_2$ are sent simultaneously, resulting in y_1, y_2. Find the capacity of this channel.

(c) Find the capacity of the following channel.



Problem 3 (Time Varying Channel)

Consider a channel with input alphabet $\mathcal{X} = \{+1, 0, 1\}$ and output alphabet $\mathcal{Y} = \{-1, 0, 1, \epsilon\}$. The channel has two states S_1 and S_2 and at each time instance it might be in wither of the states. In state S_1, the channel acts as an erasure channel with parameter ϵ, with input alphabet $\{+1, -1\}$ and output alphabet $\{-1, \epsilon, +1\}$. In state S_2, the channel acts as a Z channel with input and output alphabet $\{0, 1\}$(3).



(a) Channel in S_1 (b) Channel in S_2

Figure 1: Channel in states S_1 and S_2

The state of the channel is random process distributed according to the first order Markov process with transition probabilities shown in Fig. 3.

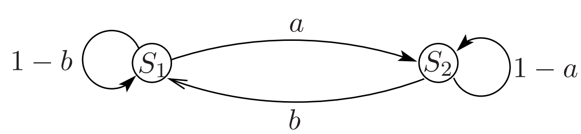


Figure 2: Transition of the states of the channel.

(a) Find the capacity of each of the channels shown in Fig. 3. What is the optimal input distribution for each of them?

(b) Compute the stationary distribution of the state of the channel.

(c) Assume that there is a genie who tells the state of the channel at each time instance to both of the encoder and decoder. Compute the capacity of the channel, C_{SI} (capacity with side information).

(d) Let the state of the channel be unknown to both encoder and decoder. What can the decoder say about the channel state?

(e) Compare the capacity of the channel when the states are unknown to the encoder/decoder, C_{NSI} (capacity without side information) to the capacity obtained in part (c). *Hint:*

Verify the following inequalities:

$$\begin{aligned} C_{NSI} &\overset{(1)}{=} \frac{1}{n} I(X^n; Y^n) \\ &\overset{(2)}{\le} \frac{1}{n} \sum_{i=1}^{n} H(X_i) - H(X_i|Y^n, S^n, X_1^{i-1}) \\ &\overset{(3)}{=} \frac{1}{n} \sum_{i=1}^{n} I(X_i; Y_i, S_i) \end{aligned}$$

Problem 4 (Channel with Memory)

Consider the discrete memoryless channel $Y_i = Z_i X_i$ with input alphabet $X_i \in \{1, 1\}$.

(a) What is the capacity of this channel when $\{Z_i\}$ is i.i.d. with

$$Z_i = \begin{cases} 1, & p = 0.5, \\ -1, & p = 0.5? \end{cases}$$

Now consider the channel with memory. Before transmission begins, Z is randomly chosen and fixed for all time. Thus, $Y_i = Z X_i$.

(b) What is the capacity if we have

$$Z = \begin{cases} 1, & p = 0.5, \\ -1, & p = 0.5? \end{cases}$$

Problem 5 (Memory Increases Capacity)

Consider a BSC with crossover probability $0 < \epsilon < 1$ represented by $X_i = Y_i + Z_i \mod 2$, where X_i, Y_i, and Z_i are, respectively, the input, the output, and the noise variable at time i. Then

$$\mathbb{P}[Z_i = 0] = 1 - \epsilon \quad \text{and} \quad \mathbb{P}[Z_i = 1] = \epsilon$$

for all i. We assume that $\{X_i\}$ and $\{Z_i\}$ are independent, but we make no assumption that Z_i are i.i.d. so that the channel may have memory.

(a) Prove that

$$I(X^n; Y^n) \le n(1 - \mathcal{H}(\mathcal{Z})).$$

(b) Show that the upper bound in (a) can be achieved using large block length n.

(c) Show that with the assumptions in (b), $I(X^n; Y^n) > nC$, where $C = 1 - h_2(\epsilon)$ is the capacity of the BSC if it is memoryless.