Quiz: Solutions

1. a) false, b) true, c) true, d) true, e) no.

2. a) \( \mathbb{E}((X + Y)^2 | Y) = \mathbb{E}(X^2) + 2 \mathbb{E}(X) Y + Y^2 \)
b) \( \mathbb{E}((X + Y)^2 | Y) = 2 \)
c) An explicit computation can be done, or one can use the following more general argument: by symmetry, \( \mathbb{E}(X|X+Y) = \mathbb{E}(Y|X+Y) \), and \( \mathbb{E}(X|X+Y) + \mathbb{E}(Y|X+Y) = \mathbb{E}(X+Y|X+Y) = X+Y \), so \( \mathbb{E}(X|X+Y) = \mathbb{E}(Y|X+Y) = \frac{X+Y}{2} \).

3. a) no, b) no, c) yes, d) \( \mathbb{E}(M_n) = 0 \), \( \text{Cov}(M_n, M_m) = 2 \) if \( m = n \), \( = 1 \) if \( |n-m| = 1 \), \( = 0 \) otherwise.

4. a) yes (apply Jensen)
b) yes: \( \mathbb{E}(S_{n+1}^4 | \mathcal{F}_n) = S_n^4 + 6S_n^2 + 1 \), so \( \mathbb{E}(S_{n+1}^4 - (n+1) | \mathcal{F}_n) = (S_n^4 - n) + 6S_n^2 \geq S_n^4 - n \).
c) From b), \( \mathbb{E}(S_{n+1}^4) = \mathbb{E}(S_n^2) + 6n + 1 \), so by induction, \( \mathbb{E}(S_n^4) = 3n^2 + 2n - 1 \)
d) \( \lim_{n \to \infty} \frac{\mathbb{E}(S_n^4)}{n^2} = 3 \). This could also be deduced directly from the central limit theorem, which states that \( S_n/\sqrt{n} \) converges in distribution to a \( \mathcal{N}(0,1) \) random variable, whose fourth moment is equal to 3.

5. a) no, b) yes, c) no, d) yes, e) no.

6. a) \( 2B_t - B_s \sim \mathcal{N}(0, 4t - 3s) \), b) yes, c) yes, d) yes, e) yes, f) no.

7. a) yes, b) NO!, c) no, d) yes, e) \( \text{Var}(M_t - M_s) = t^2 - s^2 \). NB: actually, \( M_t = \int_0^t \sqrt{2s} dB_s \)

8. \( X_t = M_t + V_t \), where \( M_t = 1 + \int_0^t aX_s dB_s \) is a martingale and \( V_t = \int_0^t (b + a^2/2) X_s ds \) is a process with bounded variation.
b) \( b + a^2/2 \geq 0 \), c) \( b + a^2/2 = 0 \), d) \( \langle X \rangle_t = a^2 \int_0^t X_s^2 ds = a^2 \int_0^t \exp(2aB_s - a^2 s) ds \).