1. a) One checks that \( \mathbb{E}(B_t^2 - t | \mathcal{F}_s^B) = B_s^2 - s \):
\[
\mathbb{E}(B_t^2 - B_s^2 | \mathcal{F}_s^B) = \mathbb{E}((B_t - B_s)^2 + 2(B_t - B_s)B_s | \mathcal{F}_s^B) = \mathbb{E}((B_t - B_s)^2) + 2\mathbb{E}(B_t - B_s)B_s = t - s.
\]
b) One checks that \( \mathbb{E}(\exp(B_t - \frac{t}{2}) | \mathcal{F}_s^B) = \exp(B_s - \frac{s}{2}) \):
\[
\mathbb{E}(\exp(B_t - B_s) | \mathcal{F}_s^B) = \mathbb{E}(\exp(B_t - B_s)) = \exp\left(\frac{t - s}{2}\right), \text{ since } B_t - B_s \sim \mathcal{N}(0, t - s) \text{ (see Ex. 4, Hw. 2)}.
\]

2. \((M_t)\) is clearly adapted to \((\mathcal{F}_t)\) and
(i) \(\mathbb{E}(|M_t|) = \mathbb{E}(|\mathbb{E}(X | \mathcal{F}_t)|) \leq \mathbb{E}(\mathbb{E}(|X| | \mathcal{F}_t)) = \mathbb{E}(|X|) < \infty\),
(ii) \(\mathbb{E}(M_t | \mathcal{F}_s) = \mathbb{E}(\mathbb{E}(X | \mathcal{F}_t) | \mathcal{F}_s) = \mathbb{E}(X | \mathcal{F}_s) = M_s\),
so \((M_t)\) is a martingale with respect to \((\mathcal{F}_t)\).

If moreover \(X\) is \(\mathcal{F}_T\)-measurable for a given \(T \in \mathbb{R}_+\), then \((M_t)\) is constant for \(t \geq T\).

3. a) For \(t > s \geq 0\), we have \(\mathbb{E}((M_t - M_s) M_s) = \mathbb{E}(\mathbb{E}(M_t - M_s | \mathcal{F}_s) M_s) = \mathbb{E}(0 M_s) = 0\).
b) For \(t > s \geq 0\), we have \(\text{Cov}(M_t, M_s) = \mathbb{E}(M_t M_s) - \mathbb{E}(M_t)\mathbb{E}(M_s) = \mathbb{E}(M_t^2) - \mathbb{E}(M_0)^2\) is only function of \(s\), therefore of \(t \wedge s\).
c) In order to determine \(c\) and \(d\), we use part (ii) of the definition of conditional expectation with \(g(y) = 1\) and \(g(y) = y\), respectively:
\[
\mathbb{E}(X1) = \mathbb{E}((cY + d) 1) = c\mathbb{E}(Y) + d, \text{ therefore } d = 0 \text{ (since } X \text{ and } Y \text{ are centered)}.
\]
\[
\mathbb{E}(XY) = \mathbb{E}((cY) Y) = c\mathbb{E}(Y^2), \text{ so } c = \frac{\mathbb{E}(X Y)}{\mathbb{E}(Y^2)}.
\]
d) \((X_t)\) is by definition adapted to its natural filtration and
(i) \(\text{For } t \geq 0, \mathbb{E}(|X_t|) < \infty, \text{ since } \mathbb{E}(X_t^2) < \infty\).
(ii) \(\text{For } t > s \geq 0: \mathbb{E}(X_t | \mathcal{F}_s^X) = \mathbb{E}(X_t | X_s) = \frac{\mathbb{E}(X_t X_s)}{\mathbb{E}(X_s^2)} X_s = X_s\) (where we have used the Markov property, part c) and the assumption).