Solutions 8

1. Remember that $B_t - B_s \sim \mathcal{N}(0, t-s)$ for $t > s \geq 0$. Therefore:

a) $B_t^{(1)} - B_s^{(1)} = -(B_t - B_s) \sim \mathcal{N}(0, t-s)$.

b) $B_t^{(2)} - B_s^{(2)} = (B_{T+t} - B_T) - (B_{T+s} - B_T) = B_{T+t} - B_{T+s} \sim \mathcal{N}(0, (T+t) - (T-s)) = \mathcal{N}(0, t-s)$.

c) $B_t^{(3)} - B_s^{(3)} = (B_T - B_{T-t}) - (B_T - B_{T-s}) = B_{T-s} - B_{T-t} \sim \mathcal{N}(0, (T-s) - (T-t)) = \mathcal{N}(0, t-s)$.

d) $B_t^{(4)} - B_s^{(4)} = \frac{1}{\sqrt{a}}(B_{at} - B_{as}) \sim \frac{1}{\sqrt{a}}\mathcal{N}(0, at - as) = \mathcal{N}(0, t-s)$.

e) $B_t^{(5)} - B_s^{(5)} = tB_{\frac{1}{t}} - sB_{\frac{1}{s}} = (t-s)B_{\frac{1}{t}} - s(B_{\frac{1}{s}} - B_{\frac{1}{t}})$, and $B_{\frac{1}{t}}$ and $(B_{\frac{1}{s}} - B_{\frac{1}{t}})$ are independent, since $t > s$. One therefore has $B_t^{(5)} - B_s^{(5)} \sim \mathcal{N}\left(0, (t-s)^2 \frac{1}{t} + s^2(\frac{1}{s} - \frac{1}{t})\right) = \mathcal{N}(0, t-s)$.

2. a) yes (i.e. $K^{(1)}$ is positive semi-definite) and $(X_t^{(1)})$ is a standard Browninan motion. The hint is easily checked for $n = 1$ or 2, and the induction gives

$$\begin{aligned} \sum_{i,j=1}^{n+1} c_i\, c_j\, (t_i \wedge t_j) &= \sum_{i,j=1}^{n} c_i\, c_j\, (t_i \wedge t_j) + 2 \sum_{1=1}^{n} c_i\, c_{n+1}\, (t_i \wedge t_{n+1}) + c_{n+1}^2\, t_{n+1} \\ &\geq \quad (c_1 + \ldots + c_n)^2\, t_{n+1} + 2 \sum_{1=1}^{n} c_i\, c_{n+1}\, t_{n+1} + c_{n+1}^2\, t_{n+1} = (c_1 + \ldots + c_{n+1})^2\, t_{n+1}. \end{aligned}$$

b) yes: $\sum_{i,j=1}^n c_i\, c_j\, g(t_i)\, g(t_j) = \left(\sum_{i=1}^n c_i\, g(t_i)\right)^2 \geq 0$ et $X_t^{(2)} = g(t)\, Y$, où $Y \sim \mathcal{N}(0,1)$.

c) no: if $(t_1, t_2) = (0, 1)$, then $\det\left(K^{(3)}(t_i, t_j)\right)_{i,j=1}^2 = \det\begin{pmatrix} 2 & 1 \\ 1 & 0 \end{pmatrix} = -1 < 0$.

d) no: if $t = 1$, then $K^{(4)}(t,t) = e^{-1} - 1 < 0$.

e) yes:

$$\sum_{i,j=1}^{n} c_i\, c_j\, (e^{t_i\, t_j} - 1) = \sum_{i,j=1}^{n} c_i\, c_j \sum_{k=1}^{\infty} \frac{(t_i\, t_j)^k}{k!} = \sum_{k=1}^{\infty} \frac{1}{k!} \left(\sum_{i=1}^{n} c_i\, t_i^k\right)^2 \geq 0.$$

3. a) $\mathbb{E}(M_t) = \mathbb{E}(B_t^2 - t) = t - t = 0$ and $\mathrm{Cov}(M_t, M_s) = 2\,(t \wedge s)^2$.

b) $\mathbb{E}(N_t) = 1$ and $\mathrm{Cov}(N_t, N_s) = e^{(t \wedge s)} - 1$.

These processes are not Gaussian, for obvious reasons.