**Solutions 4**

**1.** a) Use part (ii) of the definition with $Y \equiv 1$ ($Y$ is $\mathcal{G}$-measurable and bounded).

b) (i) $\mathbb{E}(X)$ is constant and therefore $\mathcal{G}$-measurable; (ii) Let $Y$ be $\mathcal{G}$-measurable and bounded: $\mathbb{E}(XY) = \mathbb{E}(X)\,\mathbb{E}(Y) = \mathbb{E}(\mathbb{E}(X)\,Y)$ (using the independence of $X$ and $Y$ and the linearity of expectation).

c) (i) $X$ is $\mathcal{G}$-measurable by assumption; (ii) Let $Y$ be $\mathcal{G}$-measurable and bounded: $\mathbb{E}(XY) = \mathbb{E}(XY)$!

d) (i) $\mathbb{E}(X|\mathcal{G})\,Y$ is $\mathcal{G}$-measurable; (ii) Let $Z$ be $\mathcal{G}$-measurable and bounded: $\mathbb{E}(XYZ) = \mathbb{E}(\mathbb{E}(X|\mathcal{G})\,YZ)$.

e) Let us first check the left-hand side equality: $\mathbb{E}(X|\mathcal{H})$ is $\mathcal{H}$-measurable, therefore $\mathcal{G}$-measurable, so one can apply property c).

For the right-hand side equality, one has: (i) $\mathbb{E}(X|\mathcal{H})$ is $\mathcal{H}$-measurable; (ii) Let $Y$ be $\mathcal{H}$-measurable and bounded: $\mathbb{E}(\mathbb{E}(X|\mathcal{G})\,Y) = \mathbb{E}(\mathbb{E}(XY|\mathcal{G})) = \mathbb{E}(XY) = \mathbb{E}(\mathbb{E}(X|\mathcal{H})\,Y)$ using d), a) and the definition of $\mathbb{E}(X|\mathcal{H})$.

**2.** a) One must check that $\mathbb{E}(\psi(Y)\,g(Y)) = \mathbb{E}(X\,g(Y))$ for any Borel-measurable and bounded function $g$. The computation gives

$$\mathbb{E}(\psi(Y)\,g(Y)) = \sum_{y\in C} \psi(y)\; g(y)\; \mathbb{P}(\{Y=y\}) = \sum_{x,y\in C} x\; g(y)\; \mathbb{P}(\{X=x, Y=y\}) = \mathbb{E}(X\,g(Y)).$$

b) Let $Y$ and $Z$ be the two independent dice rolls: $\mathbb{P}(\{Y=i\}) = \mathbb{P}(\{Z=j\}) = 0.25$ and $\mathbb{P}(\{Y=i, Z=j\}) = \mathbb{P}(\{Y=i\})\,\mathbb{P}(\{Z=j\})$. We therefore have $\mathbb{E}(\max(Y,Z)|Y) = \psi(Y)$, where

$$\begin{aligned}\psi(i) &= \sum_{j=i}^{4} \max(i,j)\,\mathbb{P}(\{\max(Y,Z)=j\}|\{Y=i\}) = \sum_{j=i}^{4} \max(i,j)\,\frac{\mathbb{P}(\{\max(Y,Z)=j, Y=i\})}{\mathbb{P}(\{Y=i\})} \\ &= i\,\frac{\mathbb{P}(\{Z\le i, Y=i\})}{\mathbb{P}(\{Y=i\})} + \sum_{j=i+1}^{4} j\,\frac{\mathbb{P}(\{Z=j, Y=i\})}{\mathbb{P}(\{Y=i\})} = i\,\mathbb{P}(\{Z\le i\}) + \sum_{j=i+1}^{4} j\,\mathbb{P}(\{Z=j\}).\end{aligned}$$

So $\psi(1) = 2.5$, $\psi(2) = 2.75$, $\psi(3) = 3.25$ and $\psi(4) = 4$.

**3.** a) Two possibilities for solving this question: either observe that for any Borel-measurable and bounded function $g$:

$$\mathbb{E}(\psi(Y)\,g(Y)) = \sum_{y\in C} \psi(y)\; g(y)\; \mathbb{P}(\{Y=y\}) = \sum_{x,y\in D} \varphi(x,y)\; g(y)\; \mathbb{P}(\{X=x, Y=y\}) = \mathbb{E}(\varphi(X,Y)g(Y)).$$

where the independence of $X$ and $Y$ has been used, or apply directly the formula of Exercise 2.

b) $\mathbb{E}(\max(Y,Z)) = \psi(Y)$, where $\psi(i) = \mathbb{E}(\max(i,Z)) = \sum_{j=1}^{4} \max(i,j)\,\mathbb{P}(\{Z=j\})$, which gives back the result of Exercise 2 in a simplified manner.

**4.** a) One has

$$\mathbb{P}(\{0 < X \leq t\}|\{X \geq 0,\, -\varepsilon < Y < \varepsilon\}) \;=\; \frac{\mathbb{P}(\{0 < X \leq t, -\varepsilon < Y < \varepsilon\})}{\mathbb{P}(\{X \geq 0,\, -\varepsilon < Y < \varepsilon\})}$$

$$= \frac{\int_{-\varepsilon}^{\varepsilon} dy\, \frac{1}{\pi}\, \min(t, \sqrt{1-y^2})}{\int_{-\varepsilon}^{\varepsilon} dy\, \frac{1}{\pi}\, \sqrt{1-y^2}} \underset{\varepsilon\to 0}{\sim} \frac{2\varepsilon\, t}{2\varepsilon} = t.$$

b) As $R$ and $\Theta$ are indenpendent, one has

$$\mathbb{P}(\{0 < R \leq t\}|\{-\varepsilon < \Theta < \varepsilon\}) = \mathbb{P}(\{0 < R \leq t\}) = 2\pi \int_0^t dr\, \frac{1}{\pi}\, r = r^2\Big|_0^t = t^2.$$

which does not depend on $\varepsilon$ (so remains the same in the limit $\varepsilon \to 0$).

c) The paradox is that from the above computations, one would be tempted to write:

$$\mathbb{P}(\{0 < X \leq t\}|\{X \geq 0,\, Y = 0\}) = \lim_{\varepsilon\to 0} \mathbb{P}(\{0 < X \leq t\}|\{X \geq 0,\, -\varepsilon < Y < \varepsilon\}) = t.$$

and

$$\mathbb{P}(\{0 < R \leq t\}|\{\Theta = 0\}) = \lim_{\varepsilon\to 0} \mathbb{P}(\{0 < R \leq t\}|\{-\varepsilon < \Theta < \varepsilon\}) = t^2.$$

Intuitively, the above two conditional probabilities should be the same. But conditioning on events of zero probability is forbidden. Actually, for any fixed $\varepsilon > 0$, the two events $\{-\varepsilon < Y < \varepsilon\}$ and $\{-\varepsilon < \Theta < \varepsilon\}$ are quite different: this explains why taking limits is dangerous while conditioning.