Solutions 3

1. a) Using Cauchy-Schwarz's inequality with X and $Y = 1_{\{X>t\}}$, we obtain

$$\mathbb{E}(X\,1_{\{X>t\}})^2 \leq \mathbb{E}(X^2)\,\mathbb{P}(\{X > t\}).$$

On the other hand, we have $\mathbb{E}(X\,1_{\{X>t\}}) = \mathbb{E}(X) - \mathbb{E}(X\,1_{\{X\leq t\}}) \geq \mathbb{E}(X) - t$, therefore the result.

b) We check that

$$\mathbb{P}(\{X > 0\}) = 1 - e^{-\lambda} \geq \frac{\lambda}{1+\lambda} = \frac{\mathbb{E}(X)^2}{\mathbb{E}(X^2)}.$$

(The central inequality follows from $e^{\lambda} \geq 1 + \lambda$, $\forall \lambda > 0$.)

2. a) use $\psi(x) = x^2$ and $\psi(x) = x^2 + \sigma^2$ respectively.
b) $\mathbb{P}(\{X \geq a\}) \leq \frac{\sigma^2+b^2}{(a+b)^2} = g(b)$. g has a minimum in $b = \frac{\sigma^2}{a}$ and at this point, $g(b) = \frac{\sigma^2}{a^2+\sigma^2}$.

3. Using Chebychev's inequality with $\psi(x) = x^2$, we obtain for any $\varepsilon > 0$:

$$\mathbb{P}\left(\left\{\left|\frac{S_n}{n} - \mu\right| > \varepsilon\right\}\right) = \mathbb{P}(\{|S_n - n\mu| > n\varepsilon\}) \leq \frac{\mathrm{Var}(S_n)}{(n\varepsilon)^2} = \frac{\sigma^2}{n\varepsilon^2} \underset{n\to\infty}{\to} 0.$$

where we have used:

$$\mathrm{Var}(S_n) = \sum_{i,j=1}^{n} \mathrm{Cov}(X_i, X_j) = \sum_{i=1}^{n} \mathrm{Var}(X_i) = n\sigma^2.$$

4. The sequence of gains of the player is the i.i.d. sequence $(X_1, ..., X_{369})$, with $\mathbb{P}(\{X_1 = +1\}) = \frac{18}{38}$ and $\mathbb{P}(\{X_1 = -1\}) = \frac{20}{38}$, so

$$\mu = \mathbb{E}(X_1) = \frac{18-20}{38} = -\frac{1}{19}, \quad \text{and} \quad \sigma^2 = \mathrm{Var}(X_1) = 1 - \frac{1}{361} \approx 1.$$

and the total gain after n games is $S_n = X_1 + \ldots + X_n$. We therefore obtain:

a) $\mathbb{E}(S_{361}) = 361\,\mu = -19$ francs.

b) By the central limit theorem, we have

$$\mathbb{P}(\{S_{361} > 0\}) = \mathbb{P}\left(\left\{\frac{S_{361} - 361\mu}{\sqrt{361}\sigma} > -\frac{\sqrt{361}\,\mu}{\sigma}\right\}\right) \approx \mathbb{P}\left(\left\{Z > -\frac{\sqrt{361}\mu}{\sigma}\right\}\right) \approx \mathbb{P}(\{Z > 1\}) \approx 0.15,$$

where $Z \sim \mathcal{N}(0,1)$.