Solutions 3

1. a) Using Cauchy-Schwarz’s inequality with \(X\) and \(Y = 1_{\{X > t\}}\), we obtain
\[
\mathbb{E}(X 1_{\{X > t\}})^2 \leq \mathbb{E}(X^2) \mathbb{P}(\{X > t\}).
\]
On the other hand, we have \(\mathbb{E}(X 1_{\{X > t\}}) = \mathbb{E}(X) - \mathbb{E}(X 1_{\{X \leq t\}}) \geq \mathbb{E}(X) - t\), therefore the result.

b) We check that
\[
\mathbb{P}(\{X > 0\}) = 1 - e^{-\lambda} \geq \frac{\lambda}{1 + \lambda} = \frac{\mathbb{E}(X)^2}{\mathbb{E}(X^2)}.
\]
(The central inequality follows from \(e^\lambda \geq 1 + \lambda, \forall \lambda > 0\).)

2. a) use \(\psi(x) = x^2\) and \(\psi(x) = x^2 + \sigma^2\) respectively.

b) \(\mathbb{P}(\{X \geq a\}) \leq \frac{\sigma^2 + b^2}{(a+b)^2} = g(b)\). \(g\) has a minimum in \(b = \frac{\sigma^2}{a}\) and at this point, \(g(b) = \frac{\sigma^2}{a^2 + \sigma^2}\).

3. Using Chebychev’s inequality with \(\psi(x) = x^2\), we obtain for any \(\varepsilon > 0\):
\[
\mathbb{P}\left(\left|S_n - \mu\right| > \varepsilon\right) = \mathbb{P}(\{|S_n - n\mu| > n\varepsilon\}) \leq \frac{\text{Var}(S_n)}{(n\varepsilon)^2} = \frac{\sigma^2}{n\varepsilon^2} \to 0 \text{ as } n \to \infty.
\]
where we have used:
\[
\text{Var}(S_n) = \sum_{i,j=1}^{n} \text{Cov}(X_i, X_j) = \sum_{i=1}^{n} \text{Var}(X_i) = n\sigma^2.
\]

4. The sequence of gains of the player is the i.i.d. sequence \((X_1, \ldots, X_{369})\), with \(\mathbb{P}(\{X_1 = +1\}) = \frac{18}{38}\) and \(\mathbb{P}(\{X_1 = -1\}) = \frac{20}{38}\), so
\[
\mu = \mathbb{E}(X_1) = \frac{18 - 20}{38} = -\frac{1}{19}, \quad \text{and} \quad \sigma^2 = \text{Var}(X_1) = 1 - \frac{1}{361} \approx 1.
\]
and the total gain after \(n\) games is \(S_n = X_1 + \ldots + X_n\). We therefore obtain:

a) \(\mathbb{E}(S_{361}) = 361\mu = -19\) francs.

b) By the central limit theorem, we have
\[
\mathbb{P}(\{S_{361} > 0\}) = \mathbb{P}\left(\left\{\frac{S_{361} - 361\mu}{\sqrt{361\sigma}} > -\frac{\sqrt{361}\mu}{\sigma}\right\}\right) \approx \mathbb{P}\left(\left\{Z > -\frac{\sqrt{361}\mu}{\sigma}\right\}\right) \approx \mathbb{P}(\{Z > 1\}) \approx 0.15,
\]
where \(Z \sim \mathcal{N}(0,1)\).