Solutions 2

1. a) \( \mathbb{E}(X) = p, \) \( \text{Var}(X) = p(1 - p). \)
b) \( \mathbb{E}(X) = np, \text{Var}(X) = np(1 - p). \)
c) \( \mathbb{E}(X) = \lambda, \text{Var}(X) = \lambda. \)
d) \( \mathbb{E}(X) = \frac{a+b}{2}, \text{Var}(X) = \frac{(b-a)^2}{12}. \)
e) \( \mathbb{E}(X) = \mu, \text{Var}(X) = \sigma^2. \)
f) \( \mathbb{E}(X) \) are Var(X) are not defined.
g) \( \mathbb{E}(X) = \frac{1}{\lambda}, \text{Var}(X) = \frac{1}{\lambda^2}. \)
h) \( \mathbb{E}(X) = \frac{1}{\lambda}, \text{Var}(X) = \frac{1}{\lambda^2}. \)

2. a) use \( B = A \cup (B \setminus A), \) where \( A \) and \( B \setminus A \) are disjoint.
b) use \( \cup_{n=1}^{\infty}B_n = \cup_{n=1}^{\infty}A_n, \) where \( A_n = B_n \setminus (B_1 \cup \ldots \cup B_{n-1}); \) the \( A_n \) are disjoint, so by axiom (ii) and a),
\[
\mathbb{P}(\cup_{n=1}^{\infty}B_n) = \mathbb{P}(\cup_{n=1}^{\infty}A_n) = \sum_{n=1}^{\infty} \mathbb{P}(A_n) \leq \sum_{n=1}^{\infty} \mathbb{P}(B_n)
\]
c) use again \( B = A \cup (B \setminus A). \)
d) use \( \Omega = A \cup A^c \) and \( \mathbb{P}(\Omega) = 1. \)
e) use \( A \cup B = A \cup (B \setminus (A \cap B)), \) where \( A \) and \( B \setminus (A \cap B) \) are disjoint, along with c).

3. a) We have \( \mathbb{P}(\{i\}) = 0.25 \) for all \( i, \) so \( \mathbb{P}(\{\text{red}\}) = \mathbb{P}(\{1, 4\}) = 0.5, \) \( \mathbb{P}(\{\text{odd}\}) = \mathbb{P}(\{1, 3\}) = 0.5, \) \( \mathbb{P}(\{1 \text{ or } 2\}) = \mathbb{P}(\{1, 2\}) = 0.5, \) as well as
\[
\begin{align*}
\mathbb{P}(\{\text{red} \cap \{\text{odd}\}\}) &= \mathbb{P}(\{1\}) = 0.25 = \mathbb{P}(\{\text{red}\}) \mathbb{P}(\{\text{odd}\}), \\
\mathbb{P}(\{\text{red} \cap \{1 \text{ or } 2\}\}) &= \mathbb{P}(\{1\}) = 0.25 = \mathbb{P}(\{\text{red}\}) \mathbb{P}(\{1 \text{ or } 2\}), \\
\mathbb{P}(\{\text{odd} \cap \{1 \text{ or } 2\}\}) &= \mathbb{P}(\{1\}) = 0.25 = \mathbb{P}(\{\text{odd}\}) \mathbb{P}(\{1 \text{ or } 2\}).
\end{align*}
\]
So “red”, “odd” and “1 or 2” are 2-by-2 independent, but they are not independent as a family of 3 events, since
\[
\mathbb{P}(\{\text{red} \cap \{\text{odd}\} \cap \{1 \text{ or } 2\}\}) = \mathbb{P}(\{1\}) = 0.25 \\
\neq 0.125 = \mathbb{P}(\{\text{red}\}) \mathbb{P}(\{\text{odd}\}) \mathbb{P}(\{1 \text{ or } 2\}).
\]
b) In this case, \( \mathbb{P}(\{\text{red}\}) = 0.5, \) \( \mathbb{P}(\{\text{odd}\}) = 0.5, \) \( \mathbb{P}(\{1 \text{ or } 2\}) = 0.6, \) et
\[
\begin{align*}
\mathbb{P}(\{\text{red} \cap \{\text{odd}\}\}) &= \mathbb{P}(\{1\}) = 0.3 \neq \mathbb{P}(\{\text{red}\}) \mathbb{P}(\{\text{odd}\}), \\
\mathbb{P}(\{\text{red} \cap \{1 \text{ or } 2\}\}) &= \mathbb{P}(\{1\}) = 0.3 = \mathbb{P}(\{\text{red}\}) \mathbb{P}(\{1 \text{ or } 2\}), \\
\mathbb{P}(\{\text{odd} \cap \{1 \text{ or } 2\}\}) &= \mathbb{P}(\{1\}) = 0.3 = \mathbb{P}(\{\text{odd}\}) \mathbb{P}(\{1 \text{ or } 2\}).
\end{align*}
\]
So “red” and “1 or 2”, as well as “odd” and “1 or 2” are independent, but not “red” et “odd”. From this, one deduces that the family “red”, “odd” and “1 or 2” cannot be independent.
4. a) By integration by parts, one obtains:

\[ E(X^4) = \frac{1}{\sqrt{2\pi}\sigma^2} \int_{\mathbb{R}} x^3 \cdot x \exp\left(-\frac{x^2}{2\sigma^2}\right) \, dx = \frac{1}{\sqrt{2\pi}\sigma^2} \int_{\mathbb{R}} 3x^2 \cdot \sigma^2 \exp\left(-\frac{x^2}{2\sigma^2}\right) \, dx = 3\sigma^4 \]

b) 

\[ E(\exp(X)) = \frac{1}{\sqrt{2\pi}\sigma^2} \int_{\mathbb{R}} \exp\left(x - \frac{x^2}{2\sigma^2}\right) \, dx = \frac{1}{\sqrt{2\pi}\sigma^2} \int_{\mathbb{R}} \exp\left(-\frac{(x-\sigma^2)^2}{2\sigma^2} + \frac{\sigma^2}{2}\right) \, dx = \exp\left(\frac{\sigma^2}{2}\right). \]

c) 

\[ E(\exp(-X^2)) = \frac{1}{\sqrt{2\pi}\sigma^2} \int_{\mathbb{R}} \exp\left(-x^2 \left(1 + \frac{1}{2\sigma^2}\right)\right) \, dx = \frac{1}{\sqrt{2\pi}\sigma^2} \sqrt{\frac{\pi}{1 + \frac{1}{2\sigma^2}}} = \frac{1}{\sqrt{2\sigma^2 + 1}}. \]