Solutions 12

1. a) $M_t^2 = (B_t^2 - t)^2 = f(t, B_t)$, where $f(t,x) = (x^2 - t)^2$. Notice that

$$f'_t(t,x) = -2\,(x^2 - t), \quad f'_x(t,x) = 4x\,(x^2 - t), \quad \text{and} \quad f''_{xx}(t,x) = 4\,(x^2 - t) + 8\,x^2.$$

So by Ito-Doeblin's formula, we have

$$\begin{aligned} M_t^2 - 0 &= -2\int_0^t (B_s^2 - s)\,ds + 4\int_0^t B_s\,(B_s^2 - s)\,dB_s + \frac{1}{2}\left(4\int_0^t (B_s^2 - s)\,ds + 8\int_0^t B_s^2\,ds\right) \\ &= 4\int_0^t B_s\,(B_s^2 - s)\,dB_s + 4\int_0^t B_s^2\,ds, \end{aligned}$$

and therefore $\langle M\rangle_t = 4\int_0^t B_s^2\,ds$, since the first term is a martingale. Likewise, $N_t^2 = e^{2B_t - t} = g(t, B_t)$, where $g(t,x) = e^{2x-t}$ and

$$g'_t(t,x) = -e^{2x-t}, \quad g'_x(t,x) = 2\,e^{2x-t}, \quad \text{and} \quad g''_{xx}(t,x) = 4\,e^{2x-t}.$$

So

$$N_t^2 - 1 = -\int_0^t e^{2B_s - s}\,ds + 2\int_0^t e^{2B_s - s}\,dB_s + 2\int_0^t e^{2B_s - s}\,ds = 2\int_0^t e^{2B_s - s}\,dB_s + \int_0^t e^{2B_s - s}\,ds,$$

and therefore $\langle N\rangle_t = \int_0^t e^{2B_s - s}\,ds$ since the first term is a martingale (recall that by convention, $\langle N\rangle_0 = 0$).

Remark: we could have computed $\langle M\rangle_t$ and $\langle N\rangle_t$ directly by writing M_t and N_t as stochastic integrals (using again Ito-Doeblin's formula) and using the fact that

$$\langle (H\cdot B)\rangle_t = \int_0^t H_s^2\,ds.$$

2. a) $\mathbb{E}(X_t) = 0$ and for $t \geq s$, we have

$$\begin{aligned} \mathrm{Cov}(X_t, X_s) &= \mathbb{E}(X_t X_s) = \int_0^s e^{-a(t-r)}\,e^{-a(s-r)}\,dr = e^{-a(t+s)}\int_0^s e^{2ar}\,dr \\ &= e^{-a(t+s)}\,\frac{e^{2as} - 1}{2a} = \frac{e^{-a(t-s)} - e^{-a(t+s)}}{2a}, \end{aligned}$$

so for any $t, s \geq 0$, we obtain

$$\mathrm{Cov}(X_t, X_s) = \frac{e^{-a|t-s|} - e^{-a(t+s)}}{2a} \quad \text{et} \quad \mathrm{Var}(X_t) = \frac{1 - e^{-2at}}{2a}.$$

b) The integrand $e^{-a(t-s)}$ depends both on t and s, so the process (X_t) is not a martingale and does not have independent increments.

c) Following the hint, we have $X_t = f(V_t, M_t)$, where $V_t = e^{-at} = \int_0^t (-ae^{-as})\, ds$, $M_t = \int_0^t e^{as}\, dB_s$ and $f(t,x) = tx$. Noticing that

$$f'_t(t,x) = x, \quad f'_x(t,x) = t \quad \text{and} \quad f''_{xx}(t,x) = 0,$$

we obtain

$$\begin{aligned} X_t &= f(V_t, M_t) = 0 + \int_0^t M_s\, (-ae^{-as})\, ds + \int_0^t V_s\, e^{-as}\, dB_s + 0 \\ &= -a \int_0^t M_s\, V_s\, ds + \int_0^t 1\, dB_s = -a \int_0^t X_s\, ds + B_t. \end{aligned}$$

3. a) By the optional stopping theorem, we know that $\mathbb{E}(B_T) = \mathbb{E}(B_0) = 0$, so

$$0 = \mathbb{E}(B_T) = b\, p_b + c\, p_c = b\, p_b + c\,(1 - p_b) \quad \text{and} \quad p_b = \frac{c}{c-b}.$$

Notice that $p_b \in]0,1[$, since $b < 0$ and $c > 0$. Numerical example: $p_b = \frac{1}{3}$.

b) Let us first check the hint: $f(X_t) = g(V_t, M_t)$, where $g(t,x) = f(tx)$, so

$$g'_t(t,x) = f'(tx)\, x \quad g'_x(t,x) = f'(tx)\, t \quad \text{and} \quad g''_{xx}(t,x) = f''(tx)\, t^2.$$

Therefore,

$$\begin{aligned} f(X_t) - f(X_0) &= g(V_t, M_t) - g(V_0, M_0) \\ &= \int_0^t f'(V_s M_s)\, M_s\, (-ae^{-as})\, ds + \int_0^t f'(V_s M_s)\, V_s\, e^{as}\, dB_s + \frac{1}{2} \int_0^t f''(V_s M_s)\, V_s^2 ds \\ &= \int_0^t (-aX_s f'(X_s) + \frac{1}{2} f''(X_s))\, ds + \int_0^t f'(X_s)\, dB_s = \int_0^t f'(X_s)\, dB_s, \end{aligned}$$

since f satisfies the equation of the problem set, by assumption. The process $(f(X_t))$ is therefore a martingale, so

$$f(0) = \mathbb{E}(f(X_T)) = f(b)\, p_b + f(c)\, p_c = f(b)\, p_b + f(c)\,(1 - p_b), \quad \text{so} \quad p_b = \frac{f(c) - f(0)}{f(c) - f(b)}.$$

Let us now compute the function f; $u(x) = f'(x)$ satisfies the equation:

$$-ax\, u(x) + \frac{1}{2} u'(x) = 0, \quad u(0) = 1.$$

Therefore, $\frac{u'(x)}{u(x)} = 2ax$, $\log(u(x)) = ax^2 + C$, $u(x) = \exp(ax^2 + C)$ and $C = 0$, since $u(0) = 1$. Finally, we obtain

$$f(x) = \int_0^x u(y)\, dy + D = \int_0^x \exp(ay^2)\, dy + D, \quad \text{and } D = 0 \text{ since } f(0) = 0.$$

Numerical example: by a simple drawing, we see that $p_b < \frac{1}{3}$ if $a > 0$ and $p_b > \frac{1}{3}$ if $a < 0$.