1. a) We have
\[
\text{Var}(X + Y) - \text{Var}(X - Y) = \mathbb{E}((X + Y)^2) - \mathbb{E}((X - Y)^2) - (\mathbb{E}(X + Y))^2 + (\mathbb{E}(X - Y))^2
\]
\[
= 4\mathbb{E}(XY) - 4\mathbb{E}(X)\mathbb{E}(Y) = 4\text{Cov}(X, Y).
\]

b) Using the fact that \(X_t Y_t = \frac{1}{4}((X_t + Y_t)^2 - (X_t - Y_t)^2)\), we obtain that
\[
X_t Y_t - \langle X, Y \rangle_t = \frac{1}{4}((X_t + Y_t)^2 - \langle X + Y \rangle_t) - ((X_t - Y_t)^2 - \langle X - Y \rangle_t),
\]
The process \((X_t Y_t - \langle X, Y \rangle_t)\) is therefore the difference of two martingales (by definition of the processes \((\langle X + Y \rangle_t)\) and \((\langle X - Y \rangle_t)\)); it is therefore also a martingale.

c) From the alternate definition, we observe that the process \((\langle X, Y \rangle_t)\) is the difference of two increasing processes. It is therefore a process with bounded variation.

2. a) The function \(f\) is continuous on \([0, T]\), deterministic (therefore adapted) and \(\int_0^T f(s)^2 \, ds < \infty\) (as \(f\) is continuous), so the process \((f^{(n)} \cdot B)\) is well defined. Moreover,
\[
\mathbb{E}((f^{(n)} \cdot B)^2) = \mathbb{E} \left( \int_0^t (f^{(n)}(s))^2 \, ds \right) = \int_0^t (f^{(n)}(s))^2 \, ds.
\]

b) For all \(t \in [0, T]\), \((f^{(n)} \cdot B)_t\) is a linear combination of independent Gaussian random variables; it is therefore also a Gaussian random variable. The same is true for any linear combination \(c_1 (f^{(n)} \cdot B)_{t_1} + \ldots + c_m (f^{(n)} \cdot B)_{t_m}\). The process \((f^{(n)} \cdot B)_t, t \in [0, T]\) is therefore a Gaussian process. Moreover,
\[
\mathbb{E}((f^{(n)} \cdot B)_t) = 0 \quad \text{and} \quad \text{Cov}((f^{(n)} \cdot B)_t, (f^{(n)} \cdot B)_s) = \mathbb{E}((f^{(n)} \cdot B)_t (f^{(n)} \cdot B)_s) = \int_0^{\wedge s} (f^{(n)}(r))^2 \, dr
\]

c) Since \((f^{(n)} \cdot B)\) is a martingale, we know by Exercise 3.a) in Homework 9, that it has orthogonal increments. Since \((f^{(n)} \cdot B)\) is moreover a Gaussian process, we deduce that the increments are not only orthogonal, but also independent.

3. Remark simply that
\[
\sum_{i=1}^{2^n} ((1 - \varepsilon) H_{t_{i-1}} + \varepsilon H_{t_i}) (B_{t_i} - B_{t_{i-1}})
\]
\[
= \sum_{i=1}^{2^n} H_{t_{i-1}} (B_{t_i} - B_{t_{i-1}}) + \varepsilon \sum_{i=1}^n (H_{t_i} - H_{t_{i-1}}) (B_{t_i} - B_{t_{i-1}}).
\]