Solutions 11

1. a) We have

$$\begin{aligned}\mathrm{Var}(X+Y)-\mathrm{Var}(X-Y) &= \mathbb{E}((X+Y)^2)-\mathbb{E}((X-Y)^2)-(\mathbb{E}(X+Y))^2+(\mathbb{E}(X-Y))^2\\ &= 4\mathbb{E}(XY)-4\mathbb{E}(X)\,\mathbb{E}(Y)=4\mathrm{Cov}(X,Y).\end{aligned}$$

b) Using the fact that $X_tY_t=\frac{1}{4}((X_t+Y_t)^2-(X_t-Y_t)^2)$, we obtain that

$$X_tY_t-\langle X,Y\rangle_t=\frac{1}{4}\left(\left((X_t+Y_t)^2-\langle X+Y\rangle_t\right)-\left((X_t-Y_t)^2-\langle X-Y\rangle_t\right)\right),$$

The process $(X_tY_t-\langle X,Y\rangle_t)$ is therefore the difference of two martingales (by definition of the processes $(\langle X+Y\rangle_t)$ and $(\langle X-Y\rangle_t)$); it is therfore also a martingale.

c) From the alternate definition, we observe that the process $(\langle X,Y\rangle_t)$ is the difference of two increasing processes. It is therefore a process with bounded variation.

2. a) The function f is continuous on $[0,T]$, deterministic (therefore adapted) and $\int_0^T f(s)^2\,ds<\infty$ (as f is continuous), so the process $(f^{(n)}\cdot B)$ is well defined. Moreover,

$$\mathbb{E}((f^{(n)}\cdot B)_t^2)=\mathbb{E}\left(\int_0^t (f^{(n)}(s))^2\,ds\right)=\int_0^t (f^{(n)}(s))^2\,ds.$$

b) For all $t\in[0,T]$, $(f^{(n)}\cdot B)_t$ is a linear combination of independent Gaussian random variables; it is therefore also a Gaussian random variable. The same is true for any linear combination $c_1\,(f^{(n)}\cdot B)_{t_1}+\ldots+c_m\,(f^{(n)}\cdot B)_{t_m}$. The process $((f^{(n)}\cdot B)_t,\ t\in[0,T])$ is therefore a Gaussian process. Moreover,

$$\mathbb{E}((f^{(n)}\cdot B)_t)=0\quad\text{and}\quad \mathrm{Cov}((f^{(n)}\cdot B)_t,(f^{(n)}\cdot B)_s)=\mathbb{E}((f^{(n)}\cdot B)_t\,(f^{(n)}\cdot B)_s)=\int_0^{t\wedge s}(f^{(n)}(r))^2\,dr$$

c) Since $(f^{(n)}\cdot B)$ is a martingale, we know by Exercise 3.a) in Homework 9, that it has orthogonal increments. Since $(f^{(n)}\cdot B)$ is moreover a Gaussian process, we deduce that the increments are not only orthogonal, but also independent.

3. Remark simply that

$$\begin{aligned}&\sum_{i=1}^{2^n}((1-\varepsilon)\,H_{t_{i-1}}+\varepsilon\,H_{t_i})\,(B_{t_i}-B_{t_{i-1}})\\ &\quad=\ \sum_{i=1}^{2^n}H_{t_{i-1}}\,(B_{t_i}-B_{t_{i-1}})+\varepsilon\sum_{i=1}^{n}(H_{t_i}-H_{t_{i-1}})\,(B_{t_i}-B_{t_{i-1}}).\end{aligned}$$