3 Continuous-time stochastic processes

Definition 3.1. A *continuous-time stochastic process* is a collection of random variables $(X_t,\, t \in \mathbb{R}_+)$ defined on a common probability space $(\Omega, \mathcal{F}, \mathbb{P})$. Alternatively, a stochastic process may be seen as a random function

$$X : \begin{cases} \Omega & \to & \{f : \mathbb{R}_+ \to \mathbb{R}\} \\ \omega & \mapsto & \{t \to X_t(\omega)\} \end{cases}$$

Remark. In order to describe a continuous-time stochastic process, one generally needs a LARGE probability space Ω!

Question. For a single random variable X, the knowledge of its cdf $\mathbb{P}(X \le x)$, $\forall x \in \mathbb{R}$ characterizes entirely the random variable. In the case of a stochastic process $(X_t,\, t \in \mathbb{R}_+)$, what is needed in order to characterize the process entirely?

First answer. Specify $\mathbb{P}(X_t \le x)$, $\forall t \in \mathbb{R}_+$, $\forall x \in \mathbb{R}$? This is insufficient. Here is why: assume we only know that $X_t \sim \mathcal{N}(0,t)$, $\forall t \in \mathbb{R}_+$. Let us then define

- $X_t^{(1)} = \sqrt{t}\, Y$, where $Y \sim \mathcal{N}(0,1)$.

- $X_t^{(2)}$ = standard Brownian motion (defined below).

It turns out that these two processes satisfy both $X_t^{(1)} \sim \mathcal{N}(0,t)$ and $X_t^{(2)} \sim \mathcal{N}(0,t)$, $\forall t \in \mathbb{R}_+$, even though they have little to do with each other!

Second answer. Specify $\mathbb{P}(X_{t_1} \le x_1, X_{t_2} \le x_2)$, $\forall t_1, t_2 \in \mathbb{R}_+$, $x_1, x_2 \in \mathbb{R}$? This is better, but still insufficient! (Actually, it is sufficient for Gaussian processes: see below).

n^{th} **answer.** Specify $\mathbb{P}(X_{t_1} \le x_1, \ldots, X_{t_n} \le x_n)$, $\forall t_1, \ldots, t_n \in \mathbb{R}_+$ *and* $n \ge 1$! This is the correct answer. Specifying all these joint distributions is cumbersome in general, but we will focus our attention on specific classes of processes for which a simpler description is possible.

Processes with independent and stationary increments

Definition 3.2. The random variables $X_t - X_s$, for $t \ge s \ge 0$, are called the *increments* of the process $X = (X_t,\, t \in \mathbb{R}_+)$.

Definition 3.3. A process $X = (X_t,\, t \in \mathbb{R}_+)$ is said to have *independent and stationary increments* if

- $X_t - X_s \perp\!\!\!\perp \mathcal{F}_s^X = \sigma(X_r,\, 0 \le r \le s)$, $\forall t \ge s \ge 0$ (independence).

- $X_t - X_s \sim X_{t-s} - X_0$, $\forall t \ge s \ge 0$ (stationarity).

(Remember that $X \sim Y$ means "X has the same distribution as Y").

For such process, it is sufficient to specify the distribution of X_0 and $X_t - X_0$, $\forall t \in \mathbb{R}_+$, in order to fully characterize the process. So in some sense in this case, the first answer above is valid. But having independent and stationary increments is a strong requirement for a continuous-time process.

Processes with continuous trajectories

Definition 3.4. A process $X = (X_t,\, t \in \mathbb{R}_+)$ is said to have *continuous trajectories* if

$$\mathbb{P}(\{\omega \in \Omega : \text{ the function } t \mapsto X_t(\omega) \text{ is continuous }\}) = 1.$$

We now have all the concepts in our hands in order to define the standard Brownian motion, which exhibits many interesting properties and plays a central role in the theory of stochastic calculus.

3.1 Standard Brownian motion

Definition 3.5. (first version) A *standard Brownian motion* is a continuous-time stochastic process $B = (B_t,\, t \in \mathbb{R}_+)$ such that

- $B_0 = 0$ a.s.

- B has independent and stationary increments.

- $B_t \sim \mathcal{N}(0,t)$, $\forall t \in \mathbb{R}_+$.

- B has continuous trajectories.

Basic properties. - $\mathbb{E}(B_t) = 0$, $\mathbb{E}(B_t^2) = t$, $\forall t \in \mathbb{R}_+$.

- $B_t - B_s \sim B_{t-s} - B_0 = B_{t-s} \sim \mathcal{N}(0, t-s)$, so $\mathbb{E}(B_t - B_s) = 0$, $\mathbb{E}((B_t - B_s)^2) = t - s$, $\forall t \geq s \geq 0$.

- By the law of large numbers, $\lim_{t\to\infty} \frac{B_t}{t} = 0$ a.s.

- Moreover, $\frac{B_t}{\sqrt{t}} \sim \mathcal{N}(0,1)$, $\forall t \geq 0$, so the central limit theorem applies trivially here: $\frac{B_t}{\sqrt{t}} \underset{n\to\infty}{\overset{d}{\to}} Z \sim \mathcal{N}(0,1)$, i.e.

$$\lim_{t\to\infty} \mathbb{P}\left(\frac{B_t}{\sqrt{t}} \leq x\right) = \int_{-\infty}^{x} \frac{1}{\sqrt{2\pi}} \exp(-y^2/2)\, dy, \quad \forall x \in \mathbb{R}.$$

Remarks. - These properties are reminiscent from those of the random walk.

- The existence of a process B that satisfies all the above properties is ensured by a deep and important theorem of Kolmogorov, but we shall not state it explicitly here.

Construction from the random walk.

- Let $(S_n,\, n \in \mathbb{N})$ be the simple symmetric random walk (i.e $S_0 = 0$, $S_n = \xi_1 + \ldots + \xi_n$ with ξ_i i.i.d., $\mathbb{P}(\xi_1 = 1) = \mathbb{P}(\xi_1 = -1) = \frac{1}{2}$). Remember that by the central limit theorem, $\frac{S_n}{\sqrt{n}} \underset{n\to\infty}{\overset{d}{\to}} Z \sim \mathcal{N}(0,1)$.

- Let now

$$Y_t = S_{[t]} + (t - [t])\, \xi_{[t]+1},\ t \in \mathbb{R}_+, \quad \text{i.e.,} \quad \text{if } t = n + \varepsilon,\ \varepsilon \in [0,1], \quad \text{then } Y_t = S_n + \varepsilon\, \xi_{n+1}.$$

This process is known as the *broken line process*.

Remark. Y is not a process with independent increments, nor is it a standard Brownian motion!

- Let us define $B_t^{(n)} = \frac{Y_{nt}}{\sqrt{n}}$, $t \in \mathbb{R}_+$: this amounts to looking at the process Y from far away, rescaling the x-axis by a factor n, while rescaling the y-axis by a factor $\sqrt{n}$. Assume now for simplicity that $nt \in \mathbb{N}$. Then

$$B_t^{(n)} = \frac{S_{nt}}{\sqrt{n}} = \sqrt{t}\, \frac{S_{nt}}{\sqrt{nt}} \underset{n\to\infty}{\overset{d}{\to}} \sqrt{t}\, Z \sim \mathcal{N}(0,t) \quad \text{i.e.} \quad \mathbb{P}(B_t^{(n)} \leq x) \underset{n\to\infty}{\to} \mathbb{P}(B_t \leq x).$$

as $B_t \sim \mathcal{N}(0,t)$.

- Similarly, one can show that

$$\mathbb{P}(B_{t_1}^{(n)} \leq x_1, \ldots, B_{t_m}^{(n)} \leq x_m) \underset{n\to\infty}{\to} \mathbb{P}(B_{t_1} \leq x_1, \ldots, B_{t_m} \leq x_m),$$

$\forall t_1, \ldots, t_m \in \mathbb{R}_+$, $x_1, \ldots, x_m \in \mathbb{R}$ and $m \geq 1$. This shows that the sequence of processes $B^{(n)}$ converges in distribution to the process B.

Remark. From this, we deduce that even though the limiting process B has continuous trajectories, these are nowhere differentiable. Indeed, the slope of $B_t^{(n)}$ is $\pm\sqrt{n}$, so the "slope" of B_t is $\pm\infty$. The derivative of B_t is formally called the *white noise* process (although this process does not exist!).

3.2 Mean and covariance

Let $X = (X_t,\, t \in \mathbb{R}_+)$ be a square integrable (i.e. $\mathbb{E}(X_t^2) < \infty$, $\forall t \in \mathbb{R}_+$) continuous-time process.

Definition 3.6. - The *mean* of the process X is the function $m : \mathbb{R}_+ \to \mathbb{R}$ given by $m(t) = \mathbb{E}(X_t)$, $t \in \mathbb{R}_+$.

- The *covariance* of the process X is the function $K : \mathbb{R}_+ \times \mathbb{R}_+ \to \mathbb{R}$ given by $K(t,s) = \mathrm{Cov}(X_t, X_s)$, $t, s \in \mathbb{R}_+$.

Properties. - K is symmetric, i.e. $K(t,s) = K(s,t)$.

- K is positive semi-definite, i.e.

$$\sum_{i,j=1}^n c_i\, c_j\, K(t_i, t_j) \geq 0, \quad \forall c_1, \ldots, c_n \in \mathbb{R}, \quad t_1, \ldots t_n \in \mathbb{R}_+ \quad \text{and} \quad n \geq 1.$$

The proof of this follows the same lines as the proof for the covariance of a random vector.

In general, the mean m and the covariance K alone do not fully characterize a process X (as it is the case for random variables and random vectors). The only exception is given in the following paragraph.

3.3 Gaussian processes

Definition 3.7. A *Gaussian process* is a process $(X_t,\, t \in \mathbb{R}_+)$ such that $c_1 X_{t_1} + \ldots + c_n X_{t_n}$ is a Gaussian random variable $\forall c_1, \ldots c_n \in \mathbb{R}$, $t_1, \ldots, t_n \in \mathbb{R}_+$ and $n \geq 1$.

In other words, the process X is a Gaussian process if and only if each sample $(X_{t_1}, \ldots, X_{t_n})$ is a Gaussian vector.

Theorem 3.8. (Kolmogorov) Given $m : \mathbb{R}_+ \to \mathbb{R}$ and $K : \mathbb{R}_+ \times \mathbb{R}_+ \to \mathbb{R}$ symmetric and positive semi-definite, there exists a Gaussian process X with mean m and covariance K. In addition, m and K characterize entirely the process X.

Proposition 3.9. (second possible definition of the standard Brownian motion)
The standard Brownian motion $B = (B_t,\, t \in \mathbb{R}_+)$ is a Gaussian process with continuous trajectories, with mean $m(t) = 0$ and covariance $K(t,s) = t \wedge s$ $(= \min(t,s))$.

Proof. (that the first definition implies the second)
- One should first check that $c_1 B_{t_1} + \ldots + c_n B_{t_n}$ is a Gaussian random variable $\forall c_1, \ldots, c_n \in \mathbb{R}$, $t_1, \ldots, t_n \in \mathbb{R}_+$ and $n \geq 1$. Let us simply check that $B_t + B_s$ is Gaussian $\forall t \geq s \geq 0$:

$$B_t + B_s = B_t - B_s + 2B_s \quad \text{ is Gaussian,}$$

as $B_t - B_s$ and $2B_s$ are independent and Gaussian. The proof in the general case follows the same idea.

- $m(t) = \mathbb{E}(B_t) = 0$.

- Let $t \geq s \geq 0$:

$$\begin{aligned} K(t,s) &= \mathbb{E}(B_t B_s) = \mathbb{E}((B_t - B_s + B_s) B_s) = \mathbb{E}((B_t - B_s) B_s) + \mathbb{E}(B_s^2) \\ &= \mathbb{E}(B_t - B_s)\, \mathbb{E}(B_s) + \mathbb{E}(B_s^2) = 0 + \mathbb{E}(B_s^2) = s = \min(t,s). \end{aligned}$$

□

3.4 Markov processes

Definition 3.10. A (continuous-time) *Markov process* with respect to a filtration $(\mathcal{F}_t,\, t \in \mathbb{R}_+)$ is a process $(X_t,\, t \in \mathbb{R}_+)$ such that

$$\mathbb{P}(X_t \in B \,|\, \mathcal{F}_s) = \mathbb{P}(X_t \in B \,|\, X_s) \qquad \forall t \geq s \geq 0,\, \forall B \in \mathcal{B}(\mathbb{R}).$$

Equivalently,

$$\mathbb{E}(g(X_t) \,|\, \mathcal{F}_s) = \mathbb{E}(g(X_t) \,|\, X_s) \qquad \forall t \geq s \geq 0$$

and $g : \mathbb{R} \to \mathbb{R}$ Borel-measurable and bounded.

Proposition 3.11. The standard Brownian motion is a Markov process with respect to its natural filtration $\mathcal{F}_s^B = \sigma(B_r,\, 0 \leq r \leq s)$.

Proof. - $\mathbb{E}(g(B_t) \,|\, \mathcal{F}_s^B) = \mathbb{E}(g(B_t - B_s + B_s) \,|\, \mathcal{F}_s) = \psi(B_s)$, where $\psi(y) = \mathbb{E}(g(B_t - B_s + y))$ (this follows from the fact that $B_t - B_s \perp\!\!\!\perp \mathcal{F}_s$ and that B_s is $\mathcal{F}_s$-measurable).

- Similarly, $\mathbb{E}(g(B_t) \,|\, B_s) = \mathbb{E}(g(B_t - B_s + B_s) \,|\, B_s) = \psi(B_s)$ given above. □

Remark. More generally, any process with independent increments (but not necessarily stationary) is a Markov process with respect to its natural filtration.

3.5 Martingales

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space.

Definitions 3.12. - A (continuous-time) *filtration* is a collection $(\mathcal{F}_t,\, t \in \mathbb{R}_+)$ of sub-σ-fields of $\mathcal{F}$ such that $\mathcal{F}_s \subset \mathcal{F}_t$, $\forall t \geq s \geq 0$.

- A process $(X_t,\, t \in \mathbb{R}_+)$ is said to be *adapted* to the filtration $(\mathcal{F}_t, t \in \mathbb{R}_+)$ if X_t is $\mathcal{F}_t$-measurable $\forall t \in \mathbb{R}_+$.

- The *natural filtration* of a process $(X_t,\, t \in \mathbb{R}_+)$ is defined as $\mathcal{F}_t^X = \sigma(X_s,\, 0 \leq s \leq t)$, $t \in \mathbb{R}_+$.

Remark. Every process is adapted to its natural filtration.

Definition 3.13. A process $(M_t,\, t \in \mathbb{R}_+)$ is said to be a (continuous-time) *martingale* with respect to a filtration $(\mathcal{F}_t, t \in \mathbb{R}_+)$ if

(i) $\mathbb{E}(|M_t|) < \infty$, $\forall t \in \mathbb{R}_+$.

(ii) M_t is $\mathcal{F}_t$-measurable, $\forall t \in \mathbb{R}_+$.

(iii) $\mathbb{E}(M_t \,|\, \mathcal{F}_s) = M_s$, $\forall t \geq s \geq 0$.

Generalization. The process M is said to be a *submartingale* (respectively a *supermartingale*) if condition (iii) is replaced by $\mathbb{E}(M_t \,|\, \mathcal{F}_s) \geq M_s$ (respectively $\mathbb{E}(M_t \,|\, \mathcal{F}_s) \leq M_s$), $\forall t \geq s \geq 0$.

Proposition 3.14. If $(M_t,\, t \in \mathbb{R}_+)$ is a martingale and $\varphi : \mathbb{R} \to \mathbb{R}$ is convex and such that $\mathbb{E}(|\varphi(M_t)|) < \infty$ for all $t \in \mathbb{R}_+$, then the process $(\varphi(M_t),\, t \in \mathbb{R}_+)$ is a submartingale.

Proposition 3.15. The standard Brownian motion $(B_t,\, t \in \mathbb{R}_+)$ is a martingale with respect to its natural filtration $(\mathcal{F}_t^B,\, t \in \mathbb{R}_+)$.

Proof. (i) $\mathbb{E}(|B_t|) \leq \sqrt{\mathbb{E}(B_t^2)} = \sqrt{t} < \infty$, $\forall t \in \mathbb{R}_+$.

(ii) B_t is FC_t^B-measurable, by definition, $\forall t \in \mathbb{R}_+$.

(iii) Let $t \geq s \geq 0$:

$$\mathbb{E}(B_t \,|\, \mathcal{F}_s^B) = \mathbb{E}(B_t - B_s + B_s \,|\, \mathcal{F}_s^B) = \mathbb{E}(B_t - B_s \,|\, \mathcal{F}_s^B) + \mathbb{E}(B_s \,|\, \mathcal{F}_s^B) = \mathbb{E}(B_t - B_s) + B_s = 0 + B_s = B_s.$$

□

Proposition 3.16. The following processes are also martingales with respect to $(\mathcal{F}_t^B,\, t \in \mathbb{R}_+)$:

- $(M_t = B_t^2 - t,\, t \in \mathbb{R}_+)$.

- $(N_t = \exp(B_t - \frac{t}{2}),\, t \in \mathbb{R}_+)$.

Theorem 3.17. (Lévy) (third possible definition of the standard Brownian motion)

Let $(X_t,\, t \in \mathbb{R}_+)$ be a process with continuous trajectories, adapted to a filtration $(\mathcal{F}_t,\, t \in \mathbb{R}_+)$ and such that $X_0 = 0$ a.s. and

(i) $(X_t,\, t \in \mathbb{R}_+)$ is a martingale with respect to $(\mathcal{F}_t,\, t \in \mathbb{R}_+)$.

(ii) $(X_t^2 - t,\, t \in \mathbb{R}_+)$ is also a martingale with respect to $(\mathcal{F}_t,\, t \in \mathbb{R}_+)$.

Then $(X_t,\, t \in \mathbb{R}_+)$ is a standard Brownian motion.

Definitions 3.18. - A *stopping time* with respect to a filtration $(\mathcal{F}_t,\, t \in \mathbb{R}_+)$ is a random time T with values in $\mathbb{R}_+ \cup \{+\infty\}$ such that $\{T \leq t\} \in \mathcal{F}_t$, $\forall t \in \mathbb{R}_+$.

- If X is a process, then $X_T(\omega) = X_{T(\omega)}(\omega)$, $\omega \in \Omega$ (process evaluated at time T).

- $\mathcal{F}_T = \{A \in \mathcal{F} \,:\, A \cap \{T \leq t\} \in \mathcal{F}_t,\ \forall t \in \mathbb{R}_+\}$ (information one possesses at time T).

Doob's optional sampling theorem.
Let $(M_t,\, t \in \mathbb{R}_+)$ be a martingale with respect to a filtration $(\mathcal{F}_t,\, t \in \mathbb{R}_+)$, with continuous trajectories (from now on, we will also say equivalently: a *continuous* martingale). Let T_1, T_2 be two stopping times such that $0 \leq T_1(\omega) \leq T_2(\omega) \leq K < \infty$, $\forall \omega \in \Omega$. Then $\mathbb{E}(M_{T_2} \,|\, \mathcal{F}_{T_1}) = M_{T_1}$ a.s. In particular, $\mathbb{E}(M_{T_2}) = \mathbb{E}(M_{T_1})$ (optional stopping).

Remarks. - The proof of the theorem is much more involved than in the discrete-time setting.

- The theorem remains valid for sub- and supermartingales (with corresponding inequalities).

Doob's inequalities.
Let $(M_t,\, t \in \mathbb{R}_+)$ be continuous square-integrable martingale with respect to $(\mathcal{F}_t,\, t \in \mathbb{R}_+)$ such that $M_0 = 0$ a.s. Then

a) $\mathbb{P}(\sup_{0 \leq s \leq t} |M_s| \geq \lambda) \leq \frac{\mathbb{E}(|M_t|)}{\lambda}$, $\forall t > 0$, $\lambda > 0$.

b) $\mathbb{E}(\sup_{0 \leq s \leq t} |M_s|^2) \leq 4\mathbb{E}(|M_t|^2)$, $\forall t > 0$.

Doob's decomposition theorem.
Let $(X_t,\, t \in \mathbb{R}_+)$ be a continuous submartingale with respect to a filtration $(\mathcal{F}_t,\, t \in \mathbb{R}_+)$. Then there exists a unique process $(A_t,\, t \in \mathbb{R}_+)$ which is increasing (i.e. $A_s \leq A_t$ if $s \leq t$), continuous and adapted to $(\mathcal{F}_t,\, t \in \mathbb{R}_+)$ such that $A_0 = 0$ and $(X_t - A_t,\, t \in \mathbb{R}_+)$ is a martingale with respect to $(\mathcal{F}_t,\, t \in \mathbb{R}_+)$.

Application. Let $(M_t,\, t \in \mathbb{R}_+)$ be a continuous square-integrable martingale with respect to $(\mathcal{F}_t,\, t \in \mathbb{R}_+)$. Then there exists a unique process $(A_t,\, t \in \mathbb{R}_+)$ which is increasing, continuous and adapted to $(\mathcal{F}_t,\, t \in \mathbb{R}_+)$ such that $A_0 = 0$ and $(M_t^2 - A_t,\, t \in \mathbb{R}_+)$ is a martingale with respect to $(\mathcal{F}_t,\, t \in \mathbb{R}_+)$.

This process will play a particular role in the following.

Examples. - If $M_t = B_t$, then $A_t = t$ (indeed, $B_t^2 - t$ =martingale)

- If M has independent increments, then $A_t = \mathbb{E}(M_t^2) - \mathbb{E}(M_0^2)$.

4 Stochastic integral

4.1 Functions with bounded variation

Definition 4.1. A function $g : \mathbb{R}_+ \to \mathbb{R}$ is said to have *bounded variation* if $\forall t > 0$,

$$\sup \sum_{i=1}^{n} |g(t_i) - g(t_{i-1})| < \infty,$$

where the supremum is taken over all partitions $0 = t_0 < t_1 < \ldots < t_n = t$ of $[0, t]$ (and n is arbitrary).

Examples. - If g is increasing (or decreasing), then g has bounded variation. Indeed, in this case:

$$\sum_{i=1}^{n} |g(t_i) - g(t_{i-1})| = \sum_{i=1}^{n} g(t_i) - g(t_{i-1}) = g(t_n) - g(t_0) = g(t) - g(0)$$

for all partitions of $[0, t]$, so

$$\sup \sum_{i=1}^{n} |g(t_i) - g(t_{i-1})| = g(t) - g(0) < \infty.$$

- If $g = g_1 - g_2$, where g_1 and g_2 are both increasing, then g also has bounded variation.

- If g is continuously differentiable, then g has bounded variation. Indeed,

$$\sum_{i=1}^{n} |g(t_i) - g(t_{i-1})| = \sum_{i=1}^{n} \left| \int_{t_{i-1}}^{t_i} g'(s)\, ds \right| \le \sum_{i=1}^{n} \int_{t_{i-1}}^{t_i} |g'(s)|\, ds = \int_0^t |g'(s)|\, ds < \infty.$$

Again, this expression does not depend on the chosen partition, so

$$\sup \sum_{i=1}^{n} |g(t_i) - g(t_{i-1})| \le \int_0^t |g'(s)|\, ds < \infty.$$

Generalization to processes.

Definition 4.2. A continuous-time stochastic process $(X_t,\, t \in \mathbb{R}_+)$ is said to have bounded variation if its trajectories have bounded variation a.s.

We will see that the trajectories of the standard Brownian motion have *unbounded variation*, a.s.

4.2 Quadratic variation

Quadratic variation of the standard Brownian motion.

Let $(B_t,\, t \in \mathbb{R}_+)$ be a standard Brownian motion. For $t > 0$ and $n \ge 1$ fixed, let

$$\langle B \rangle_t^{(n)} = \sum_{i=1}^{2^n} \left(B\left(\frac{it}{2^n}\right) - B\left(\frac{(i-1)t}{2^n}\right)\right)^2.$$

Notation. We use indifferently the notation $B_t \equiv B(t)$.

Definition 4.3. The (almost sure) limit $\langle B \rangle_t = \lim_{n\to\infty} \langle B \rangle_t^{(n)}$ is called the *quadratic variation* of the Brownian motion. We show below that it exists and is equal to t.

Proposition 4.4. For every fixed $t \geq 0$, $\langle B \rangle_t = t$, a.s.

Proof. Recall that in order to show that $Z_n \to Z$ a.s., it is sufficient to check that

$$\sum_{n\geq 1} \mathbb{P}(\{|Z_N - Z| > \varepsilon\}) < \infty, \quad \forall \varepsilon > 0.$$

Here, $Z_n = \langle B \rangle_t^{(n)}$ and $Z = t$, which is fixed. Let us first compute $\mathbb{E}(\langle B \rangle_t^{(n)})$ and $\mathrm{Var}(\langle B \rangle_t^{(n)})$:

$$\mathbb{E}(\langle B \rangle_t^{(n)}) = \sum_{i=1}^{2^n} \mathbb{E}\Big(\Big(\underbrace{B\left(\tfrac{it}{2^n}\right) - B\left(\tfrac{(i-1)t}{2^n}\right)}_{\sim \mathcal{N}(0,\frac{t}{2^n})}\Big)^2\Big) = \sum_{i=1}^{2^n} \frac{t}{2^n} = t$$

and

$$\mathrm{Var}(\langle B \rangle_t^{(n)}) = \sum_{i=1}^{2^n} \mathrm{Var}\Big(\Big(B\left(\tfrac{it}{2^n}\right) - B\left(\tfrac{(i-1)t}{2^n}\right)\Big)^2\Big)$$

by independence of the increments of B. Moreover, if $X \sim \mathcal{N}(0, \sigma^2)$, then

$$\mathrm{Var}(X^2) = \mathbb{E}(X^4) - \mathbb{E}(X^2)^2 = 3\sigma^4 - \sigma^4 = 2\sigma^4, \quad \text{so} \quad \mathrm{Var}(\langle B \rangle_t^{(n)}) = \sum_{i=1}^{2^n} 2\left(\tfrac{t}{2^n}\right)^2 = \frac{t^2}{2^{n-1}}.$$

Therefore, by Chebychev's inequality,

$$\mathbb{P}(\{|\langle B \rangle_t^{(n)} - t| > \varepsilon\}) \leq \frac{1}{\varepsilon^2}\,\mathbb{E}((\langle B \rangle_t^{(n)} - t)^2) = \frac{1}{\varepsilon^2}\,\mathrm{Var}(\langle B \rangle_t^{(n)}) = \frac{t^2}{\varepsilon^2\, 2^{n-1}}$$

and

$$\sum_{n\geq 1} \mathbb{P}(\{|\langle B \rangle_t^{(n)} - t| > \varepsilon\}) \leq \frac{t^2}{\varepsilon^2} \underbrace{\sum_{n\geq 1} \frac{1}{2^{n-1}}}_{=1} < \infty, \quad \forall \varepsilon > 0,$$

so the proposition is proved. □

Corollary 4.5. For all $t > 0$, we have

$$\lim_{n\to\infty} \sum_{i=1}^{2^n} \left| B\left(\frac{it}{2^n}\right) - B\left(\frac{(i-1)t}{2^n}\right) \right| = \infty \quad \text{a.s.}$$

Consequently, the process $(B_t,\, t \in \mathbb{R}_+)$ has unbounded variation, a.s.

Proof. Let us first check that if $g : \mathbb{R}_+ \to \mathbb{R}$ is a continuous function such that

$$\lim_{n\to\infty} \sum_{i=1}^{2^n} \left| g\left(\frac{it}{2^n}\right) - g\left(\frac{(i-1)t}{2^n}\right) \right| < \infty, \quad \text{then} \quad \langle g \rangle_t = \lim_{n\to\infty} \sum_{i=1}^{2^n} \left(g\left(\frac{it}{2^n}\right) - g\left(\frac{(i-1)t}{2^n}\right) \right)^2 = 0.$$

Indeed,

$$\begin{aligned} &\lim_{n\to\infty} \sum_{i=1}^{2^n} \left(g\left(\frac{it}{2^n}\right) - g\left(\frac{(i-1)t}{2^n}\right) \right)^2 \\ \leq\; &\underbrace{\lim_{n\to\infty} \max_{1\leq i\leq 2^n} \left| g\left(\frac{it}{2^n}\right) - g\left(\frac{(i-1)t}{2^n}\right) \right|}_{=0} \cdot \underbrace{\lim_{n\to\infty} \sum_{i=1}^{2^n} \left| g\left(\frac{it}{2^n}\right) - g\left(\frac{(i-1)t}{2^n}\right) \right|}_{<\infty} = 0. \end{aligned}$$

So, as we know that the Brownian motion B has continuous trajectories, if it was the case that

$$\mathbb{P}\left(\lim_{n\to\infty}\sum_{i=1}^{2^n}\left|B\left(\frac{it}{2^n}\right)-B\left(\frac{(i-1)t}{2^n}\right)\right|<\infty\right)>0,$$

then this would imply that $\mathbb{P}(\langle B\rangle_t=0)>0$, which is in contradiction with the previous result ($\langle B\rangle_t=t$ a.s.). In conclusion,

$$\mathbb{P}\left(\lim_{n\to\infty}\sum_{i=1}^{2^n}\left|B\left(\frac{it}{2^n}\right)-B\left(\frac{(i-1)t}{2^n}\right)\right|=\infty\right)=1.$$

□

Final remark. Notice that $B_t^2-\langle B\rangle_t=B_t^2-t$ is a martingale. This is not a coincidence.

Quadratic variation of a martingale.

Reminder(from Doob's decomposition theorem). If $(M_t,\,t\in\mathbb{R}_+)$ is a continuous square-integrable martingale, then there exists a unique process $(A_t,\,t\in\mathbb{R}_+)$ which is increasing, continuous and adapted to the same filtration as $(M_t,\,t\in\mathbb{R}_+)$, such that $A_0=0$ and $(M_t^2-A_t,\,t\in\mathbb{R}_+)$ is a martingale.

Definition 4.6. The process A is called the *quadratic variation* of the martingale M and is denoted as $A_t=\langle M\rangle_t$, $t\in\mathbb{R}_+$.

Proposition 4.7. If $(M_t,\,t\in\mathbb{R}_+)$ is a continuous square-integrable martingale, then

$$\langle M\rangle_t^{(n)}=\sum_{i=1}^{2^n}\left(M\left(\frac{it}{2^n}\right)-M\left(\frac{(i-1)t}{2^n}\right)\right)^2\underset{n\to\infty}{\overset{\mathbb{P}}{\to}}\langle M\rangle_t,\quad\forall t>0,$$

where $(\langle M\rangle_t,\,t\in\mathbb{R}_+)$ is the process defined above.

Remarks. - By the above definition, $\mathbb{E}(\langle M\rangle_t)=\mathbb{E}(M_t^2)-\mathbb{E}(M_0^2)$.

- The process $\langle M\rangle$ is increasing : it therefore has bounded variation itself.

- The only martingales with quadratic variation equal to zero are constant processes! So all non-constant martingales have unbounded variation!

Quadratic covariation.

Let M,N be two continuous square-integrable martingales (adapted to the same filtration $(\mathcal{F}_t,\,t\in\mathbb{R}_+)$).

Definition 4.8. The *quadratic covariation* of M and N is the unique process $\langle M,N\rangle$ which is continuous, adapted, has bounded variation and is such that $\langle M,N\rangle_0=0$ and $(M_tN_t-\langle M,N\rangle_t,\,t\in\mathbb{R}_+)$ is a martingale.

Remark. $\langle M,M\rangle_t=\langle M\rangle_t$.

Proposition 4.9.

$$\langle M,N\rangle_t^{(n)}=\sum_{i=1}^{2^n}\left(M\left(\frac{it}{2^n}\right)-M\left(\frac{(i-1)t}{2^n}\right)\right)\left(N\left(\frac{it}{2^n}\right)-N\left(\frac{(i-1)t}{2^n}\right)\right)\underset{n\to\infty}{\overset{\mathbb{P}}{\to}}\langle M,N\rangle_t,\quad\forall t>0.$$

Proposition 4.10. If $c\in\mathbb{R}$ and M, N are independent, then for all $t\in\mathbb{R}_+$,

$$\langle M,N\rangle_t=0\quad\text{and}\quad\langle cM+N\rangle_t=c^2\langle M\rangle_t+\langle N\rangle_t.$$

Remark. From the above two propositions, we see that the quadratic variation of a martingale plays the same role as the variance of a random variable. Likewise, the quadratic covariation of two martingales plays the same role as the covariance of two random variables.