Homework 9

Exercise 1. Let $(B_t,\, t \in \mathbb{R}_+)$ be a standard Brownian motion. Let us also denote by $(\mathcal{F}_t^B,\, t \in \mathbb{R}_+)$ its natural filtration. Show that the two processes below (already defined in Hw. 8, Ex. 3) are martingales with respect to $(\mathcal{F}_t^B)$:

a) $M_t = B_t^2 - t, \quad t \in \mathbb{R}_+$.

b) $N_t = \exp(B_t - \frac{t}{2}), \quad t \in \mathbb{R}_+$.

Exercise 2. Let X be a random variable such that $\mathbb{E}(|X|) < \infty$ and $(\mathcal{F}_t, t \in \mathbb{R}_+)$ be a filtration. Show that the process $(M_t,\, t \in \mathbb{R}_+)$ defined as $M_t = \mathbb{E}(X|\mathcal{F}_t)$ is a martingale with respect to $(\mathcal{F}_t)$. What can be said about the process (M_t) when X is an $\mathcal{F}_T$-measurable random variable for a given $T \in \mathbb{R}_+$?

Exercise 3. a) Let $(M_t, t \in \mathbb{R}_+)$ be a square-integrable continuous martingale. Show that (M_t) is a process with *orthogonal increments*, that is:

$$\mathbb{E}((M_t - M_s)\, M_s) = 0, \quad \forall t > s \geq 0.$$

b) Deduce that $\mathrm{Cov}(M_t, M_s)$ depends on $t \wedge s$ only.

Hint: We already know that $\mathbb{E}(M_t)$ is constant over time.

c) Let (X, Y) be a centered Gaussian vector. Show that

$$\mathbb{E}(X|Y) = \frac{\mathbb{E}(XY)}{\mathbb{E}(Y^2)}\, Y \text{ a.s.}$$

Hint: We will admit here that we already know that the conditional expectation $\mathbb{E}(X|Y)$ is of the form $cY + d$ when (X, Y) is a Gaussian vector.

d) Let $(X_t, t \in \mathbb{R}_+)$ be a centered Gaussian process, verifying the Markov property and such that

$$\mathbb{E}((X_t - X_s)\, X_s) = 0, \quad \forall t > s \geq 0.$$

Show that (X_t) is a martingale with respect to its natural filtration $(\mathcal{F}_t^X)$.