Exercise 1. Let \((B_t, t \in \mathbb{R}_+)\) be a standard Brownian motion. Let us also denote by \((\mathcal{F}^B_t, t \in \mathbb{R}_+)\) its natural filtration. Show that the two processes below (already defined in Hw. 8, Ex. 3) are martingales with respect to \((\mathcal{F}^B_t)\):

a) \(M_t = B_t^2 - t, \quad t \in \mathbb{R}_+\).

b) \(N_t = \exp(B_t - \frac{t^2}{2}), \quad t \in \mathbb{R}_+\).

Exercise 2. Let \(X\) be a random variable such that \(\mathbb{E}(|X|) < \infty\) and \((\mathcal{F}_t, t \in \mathbb{R}_+)\) be a filtration. Show that the process \((M_t, t \in \mathbb{R}_+)\) defined as \(M_t = \mathbb{E}(X | \mathcal{F}_t)\) is a martingale with respect to \((\mathcal{F}_t)\). What can be said about the process \((M_t)\) when \(X\) is an \(\mathcal{F}_T\)-measurable random variable for a given \(T \in \mathbb{R}_+\)?

Exercise 3. a) Let \((M_t, t \in \mathbb{R}_+)\) be a square-integrable continuous martingale. Show that \((M_t)\) is a process with orthogonal increments, that is:

\[
\mathbb{E}((M_t - M_s) M_s) = 0, \quad \forall t > s \geq 0.
\]

b) Deduce that \(\text{Cov}(M_t, M_s)\) depends on \(t \wedge s\) only.

Hint: We already know that \(\mathbb{E}(M_t)\) is constant over time.

c) Let \((X, Y)\) be a centered Gaussian vector. Show that 

\[
\mathbb{E}(X|Y) = \frac{\mathbb{E}(XY)}{\mathbb{E}(Y^2)} Y \text{ a.s.}
\]

Hint: We will admit here that we already know that the conditional expectation \(\mathbb{E}(X|Y)\) is of the form \(cY + d\) when \((X, Y)\) is a Gaussian vector.

d) Let \((X_t, t \in \mathbb{R}_+)\) be a centered Gaussian process, verifying the Markov property and such that 

\[
\mathbb{E}((X_t - X_s) X_s) = 0, \quad \forall t > s \geq 0.
\]

Show that \((X_t)\) is a martingale with respect to its natural filtration \((\mathcal{F}^X_t)\).