Exercise 1. Let \((B_t, t \in \mathbb{R}_+)\) be a standard Brownian motion. Check that the increments of the five processes below are again distributed as those of a standard Brownian motion:

a) \(B_t^{(1)} = -B_t, t \in \mathbb{R}_+\) (\(\leftrightarrow\) “spatial” symmetry of the Brownian motion).

b) Let \(T \in \mathbb{R}_+: B_t^{(2)} = B_{t+T} - B_T, t \in \mathbb{R}_+\) (\(\leftrightarrow\) stationarity).

c) Let \(T \in \mathbb{R}_+: B_t^{(3)} = B_T - B_{T-t}, t \in [0, T]\) (\(\leftrightarrow\) time-reversal).

d) Let \(a > 0: B_t^{(4)} = \frac{1}{\sqrt{a}} B_{at}, t \in \mathbb{R}_+\) (\(\leftrightarrow\) scaling law).

e) \(B_t^{(5)} = tB_{\frac{1}{t}}, t > 0\) and \(B_0^{(5)} = 0\) (\(\leftrightarrow\) time inversion).

Remark: These five processes are actually all standard Brownian motions (proof not required).

Exercise 2. Among the symmetric functions \(K: \mathbb{R}_+ \times \mathbb{R}_+ \rightarrow \mathbb{R}\) below, determine which are positive semi-definite. When this is the case, describe the centered Gaussian process \((X_t, t \in \mathbb{R}_+)\) with covariance \(K(t, s) = \mathbb{E}(X_t X_s)\).

a) \(K^{(1)}(t, s) = t \wedge s[= \min\{t, s\}]\).

Hint: Show by induction that if \(t_1 \geq \ldots \geq t_n\), then \(\sum_{i,j=1}^{n} c_i c_j (t_i \wedge t_j) \geq (c_1 + \ldots + c_n)^2 t_n \geq 0\).

b) \(K^{(2)}(t, s) = g(t) g(s), \) where \(g: \mathbb{R}_+ \rightarrow \mathbb{R}\) is continuous.

c) \(K^{(3)}(t, s) = t + s\).

d) \(K^{(4)}(t, s) = e^{-ts} - 1\).

e) \(K^{(5)}(t, s) = e^{ts} - 1\).

Remark: For the last two cases, do not try describing the Gaussian process with covariance \(K(t, s)\) (if it exists!).

Exercise 3. Let \((B_t, t \in \mathbb{R}_+)\) be a standard Brownian motion. Compute the mean and the covariance of the following two processes:

a) \(M_t = B_t^2 - t, \quad t \in \mathbb{R}_+\).

b) \(N_t = \exp(B_t - \frac{t}{2}), \quad t \in \mathbb{R}_+\).

Is any of these two processes Gaussian?