Exercise 1. Let \((\xi_n, n \geq 1)\) be a sequence of i.i.d. centered random variables and let \((\mathcal{F}_n, n \geq 1)\) be the filtration defined as \(\mathcal{F}_n = \sigma(\xi_1, \ldots, \xi_n), n \geq 1\). Among the following processes \((X_n, n \geq 1)\), which are Markov processes with respect to \((\mathcal{F}_n, n \geq 1)\)? which are martingales with respect to \((\mathcal{F}_n, n \geq 1)\)? (no formal justification needed; the answer suffices)

a) \(X_n = \xi_n, n \geq 1\).

b) \(X_1 = \xi_1, X_{n+1} = aX_n + \xi_{n+1}, n \geq 1\) (\(a > 0\) fixed).

c) \(X_1 = \xi_1, X_{n+1} = \xi_n + \xi_{n+1}, n \geq 1\).

d) \(X_n = \max(\xi_1, \ldots, \xi_n), n \geq 1\).

e) \(X_1 = \xi_1, X_n = \sum_{i=1}^{n}(\xi_1 + \cdots + \xi_{i-1})\xi_i, n \geq 1\).

Exercise 2. An urban legend says: “two uncorrelated Gaussian random variables are necessarily independent”. The exercise below shows that this is wrong.

Let \(X, Y\) be two centered Gaussian random variables, with variance 1.

a) Show that if \(X\) and \(Y\) are independent, then \((X, Y)\) is a Gaussian vector, so \(X + Y\) is Gaussian.

b) Let \(E\) be a random variable independent of \(X\) and such that \(\mathbb{P}(E = +1) = \mathbb{P}(E = -1) = \frac{1}{2}\). Show that \(Z = E X\) is Gaussian, but that \(X + Z\) is not, and therefore that \((X, Z)\) is not a Gaussian vector.

c) Show also that \(X\) and \(Z\) are not independent, even though \(\text{Cov}(X, Z) = 0\).

Exercise 3. Let \(a \in \mathbb{R}\) and \(X = (X_1, X_2, X_3)\) be a centered random vector (not necessarily Gaussian) such that

\[
\mathbb{E}(X_1^2) = \mathbb{E}(X_2^2) = \mathbb{E}(X_3^2) = 1 \quad \text{and} \quad \mathbb{E}(X_1 X_2) = \mathbb{E}(X_2 X_3) = \mathbb{E}(X_1 X_3) = a.
\]

What values are allowed for the parameter \(a\)? In particular, can it be that \(\mathbb{E}(X_1 X_2) = \mathbb{E}(X_2 X_3) = \mathbb{E}(X_1 X_3) = -1\)?

Hint: Compute the eigenvalues of the covariance matrix of \(X\).

Exercise 4. a) Let us assume that the vector \(X\) defined in Exercise 3 is Gaussian. For what values of \(a\) is the vector \(X\) degenerate? Describe the vector \(X\) in these cases.

b) Let \(b \in [-1, +1]\) and \(Y = (Y_1, Y_2, Y_3)\) be a centered Gaussian random vector whose covariance matrix is given by

\[
K = \begin{pmatrix}
1 & b & b^2 \\
b & 1 & b \\
b^2 & b & 1
\end{pmatrix}
\]

For what values of \(b\) is the vector \(Y\) degenerate? Describe the vector \(Y\) in these cases.