Homework 6

Exercise 1. a) Let $(X_n,\, n \in \mathbb{N})$ be the simple asymmetric random walk, that is: $X_0 = 0$ and $X_n = \xi_1 + ... + \xi_n$, where $\xi_1, ..., \xi_n$ are i.i.d. random variables such that $\mathbb{P}(\xi_1 = +1) = p = 1 - \mathbb{P}(\xi_1 = -1)$, with $p > 1/2$.

Find the Doob decomposition of the submartingale $(X_n,\, n \in \mathbb{N})$, i.e. find the processes $(M_n,\, n \in \mathbb{N})$ and $(A_n,\, n \in \mathbb{N})$ such that $(M_n,\, n \in \mathbb{N})$ is a martingale, $(A_n,\, n \in \mathbb{N})$ is an increasing and predictable process such that $A_0 = 0$ a.s. and $X_n = M_n + A_n$, for all $n \in \mathbb{N}$.

b) Let now $X_n = S_n^2$, where $(S_n,\, n \in \mathbb{N})$ is the simple *symmetric* random walk on $\mathbb{Z}$.

Find again the Doob decomposition of the submartingale $(X_n,\, n \in \mathbb{N})$.

Exercise 2. Let $(S_n,\, n \in \mathbb{N})$ be the simple symmetric random walk and $(\mathcal{F}_n,\, n \in \mathbb{N})$ be its natural filtration. Among the following random times, which are stopping times with respect to $(\mathcal{F}_n,\, n \in \mathbb{N})$? which are bounded? (no formal justification needed: the answer suffices)

a) $T = \sup\{n \geq 0 \,:\, S_n \geq a\}$ ($a > 0$ is fixed)

b) $T = \inf\{n \geq 1 \,:\, S_n = \max_{0 \leq k \leq n} S_k\}$

c) $T = \inf\{n \geq 0 \,:\, S_n = \max_{0 \leq m \leq N} S_m\}$ ($N \geq 1$ is fixed)

d) $T = \inf\{n \geq 0 \,:\, S_n \geq a \text{ or } n \geq N\}$ ($a > 0$ and $N \geq 1$ are fixed)

Exercise 3. Let $(S_n,\, n \in \mathbb{N})$ be the simple symmetric simple random walk, $(\mathcal{F}_n,\, n \in \mathbb{N})$ be its natural filtration and

$$T = \inf\{n \geq 1 \,:\, |S_n| \geq a\}$$

where $a \geq 1$ is an integer number.

a) Show that T is a stopping time with respect to $(\mathcal{F}_n,\, n \in \mathbb{N})$.

Let now $(M_n,\, n \in \mathbb{N})$ be defined as $M_n = S_n^2 - n$, for all $n \in \mathbb{N}$. The process $(M_n,\, n \in \mathbb{N})$ has been shown to be a martingale with respect to $(\mathcal{F}_n,\, n \in \mathbb{N})$ in Homework 5, Exercise 3.

b) Apply the optional stopping theorem to compute $\mathbb{E}(T)$.

Remark: Even though T is an unbounded stopping time, the optional stopping theorem applies here. Notice that the theorem would *not* apply if one would consider the following stopping time:

$$T' = \inf\{n \geq 1 \,:\, S_n \geq a\}.$$

We have seen a similar example of that in the class.

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Exercise 4. (If one cannot win on a game, then it is a martingale)
Let $(\mathcal{F}_n,\, n \in \mathbb{N})$ be a filtration and $(M_n,\, n \in \mathbb{N})$ be a process adapted to $(\mathcal{F}_n,\, n \in \mathbb{N})$ such that $\mathbb{E}(|M_n|) < \infty$, for all $n \in \mathbb{N}$.

Show that if for any predictable process $(H_n,\, n \in \mathbb{N})$ such that H_n is a bounded random variable $\forall n \in \mathbb{N}$, we have

$$\mathbb{E}((H \cdot M)_N) = 0, \quad \forall N \in \mathbb{N},$$

then $(M_n,\, n \in \mathbb{N})$ is a martingale with respect to $(\mathcal{F}_n,\, n \in \mathbb{N})$.