Homework 5

Exercise 1. Let $(M_n,\, n \in \mathbb{N})$ be a martingale with respect to the filtration $(\mathcal{F}_n,\, n \in \mathbb{N})$. Show that

a) $\mathbb{E}(M_{n+1}) = \mathbb{E}(M_n)$, $\forall n \in \mathbb{N}$.

b) $\mathbb{E}(M_{n+1} - M_n | \mathcal{F}_n) = 0$ a.s., $\forall n \in \mathbb{N}$.

c) $\mathbb{E}(M_{n+m} | \mathcal{F}_n) = M_n$ a.s., $\forall m, n \in \mathbb{N}$.

Exercise 2. Let $(X_n,\, n \in \mathbb{N})$ be a submartingale and $\varphi : \mathbb{R} \to \mathbb{R}$ be a Borel-measurable and convex function such that $\mathbb{E}(|\varphi(X_n)|) < \infty$, $\forall n \in \mathbb{N}$.

a) What additional property of φ ensures that the process $(\varphi(X_n),\, n \in \mathbb{N})$ is also a submartingale?

b) In particular, which of the following processes is ensured to be a submartingale: $(X_n^2,\, n \in \mathbb{N})$ or $(\exp(X_n),\, n \in \mathbb{N})$?

Exercise 3. Let $(S_n,\, n \in \mathbb{N})$ be the simple symmetric random walk on $\mathbb{Z}$, that is: $S_0 = 0$, $S_n = \xi_1 + ... + \xi_n$, where $\xi_1, \ldots, \xi_n$ are i.i.d. random variables such that $\mathbb{P}(\xi_1 = +1) = \mathbb{P}(\xi_1 = -1) = 1/2$. Let also $\mathcal{F}_0 = \{\emptyset, \Omega\}$ and $\mathcal{F}_n = \sigma(\xi_1, \ldots, \xi_n)$ for $n \geq 1$.

a) Show that $(\mathcal{F}_n,\, n \in \mathbb{N})$ is the natural filtration of the process $(S_n,\, n \in \mathbb{N})$.

b) Show that the process $(S_n^2 - n,\, n \in \mathbb{N})$ is a martingale with respect to $(\mathcal{F}_n,\, n \in \mathbb{N})$.

Exercise 4. ("The" martingale)
A player bets on a sequence of i.i.d. (and balanced) coin tosses: at each turn, the player wins twice his bet if the coin falls on "heads" or loses his bet if the coin falls on "tails".

Assume now that the player adopts the following strategy: he starts by betting 1 franc. If he wins his bet (that is, if the outcome is "heads"), he quits the game and does not bet anymore. If he loses (that is, if the outcome is "tails"), he plays again and doubles his bet for the next turn. He then goes on with the same strategy for the rest of the game.

We assume here that the player can borrow any money he wants in order to bet. Of course, we also assume that he has no information on the outcome of the next coin toss while betting on it.

a) Is the process of gains of the player a martingale (by convention, we set the gain of the player at time zero to be equal to zero)?

b) What is the gain of the player at the first time "heads" comes out?

c) Isn't there a contradiction between a) and b)?