Homework 4

Exercise 1. Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space, X be an integrable random variable defined on this space and let $\mathcal{G}$ be a sub-σ-field of $\mathcal{F}$. Relying only on the definition of conditional expectation, show the following properties:

a) $\mathbb{E}(\mathbb{E}(X|\mathcal{G})) = \mathbb{E}(X)$.

b) If X is independent of $\mathcal{G}$, then $\mathbb{E}(X|\mathcal{G}) = \mathbb{E}(X)$ a.s.

c) If X is $\mathcal{G}$-measurable, then $\mathbb{E}(X|\mathcal{G}) = X$ a.s.

d) If Y is $\mathcal{G}$-measurable et bounded, then $\mathbb{E}(XY|\mathcal{G}) = \mathbb{E}(X|\mathcal{G})\, Y$ a.s.

e) If $\mathcal{H}$ is a sub-σ-field of $\mathcal{G}$, then $\mathbb{E}(\mathbb{E}(X|\mathcal{H})|\mathcal{G}) = \mathbb{E}(X|\mathcal{H}) = \mathbb{E}(\mathbb{E}(X|\mathcal{G})|\mathcal{H})$ a.s.

Exercise 2. Let X, Y be two discrete random variables (with values in a countable set C). Let us moreover assume that X is integrable.

a) Show that the random variable $\psi(Y)$, where ψ is defined as

$$\psi(y) = \sum_{x \in C} x\, \mathbb{P}(\{X = x\}|\{Y = y\}),$$

matches the theoretical definition of conditional expectation $\mathbb{E}(X|Y)$ given in class.

b) *Application:* One rolls two independent and balanced dice (say Y and Z), each with four faces. What is the conditional expectation of the maximum of the two, given the value of one of them?

Exercise 3. Let X, Y be two independent discrete random variables and $\varphi : \mathbb{R}^2 \to \mathbb{R}$ be a Borel-measurable function such that $\mathbb{E}(|\varphi(X, Y)|) < \infty$.

a) Show that

$$\mathbb{E}(\varphi(X, Y)|Y) = \psi(Y), \quad \text{where} \quad \psi(y) = \mathbb{E}(\varphi(X, y)).$$

b) Reconsider the application of Exercise 2 with this formula.

please turn the page %

Exercise 4. (Borel's paradox)
Let Z be a two-dimensional random variable uniformly distributed on the unit disc $B(0,1)$ in $\mathbb{R}^2$. Z has two possible representations:

(i) $Z = (X, Y)$, where $X \in [-1, 1]$ and $Y \in [-1, 1]$ are the horizontal and vertical coordinates of Z respectively, with joint pdf

$$f_{X,Y}(x, y) = \frac{1}{\pi} 1_{x^2+y^2 \leq 1}.$$

(ii) $Z = (R, \Theta)$, where $R \in [0, 1]$ is the radius of Z and $\Theta \in]-\pi, \pi]$ is its angle with respect to the horizontal axis. Their joint pdf is given by

$$f_{R,\Theta}(r, \theta) = \frac{1}{\pi} r \, 1_{0 \leq r \leq 1} \, 1_{-\pi < \theta \leq \pi},$$

where the factor r comes from the Jacobian of the change of coordinates.

a) For $t \in [0, 1]$, compute $\lim_{\varepsilon \to 0} \mathbb{P}(\{0 < X \leq t\} | \{X \geq 0, -\varepsilon < Y < \varepsilon\})$.

b) For $t \in [0, 1]$, compute $\lim_{\varepsilon \to 0} \mathbb{P}(\{0 < R \leq t\} | \{-\varepsilon < \Theta < \varepsilon\})$.

c) What is the paradox here? Can you resolve it?