Homework 10

Exercise 1. The *quartic* variation of a standard Brownian motion $(B_t,\, t \in \mathbb{R}_+)$ on a fixed interval $[0,t]$ is given by the following limit:

$$\lim_{n\to\infty} \sum_{i=1}^{2^n} \left(B\left(\frac{it}{2^n}\right) - B\left(\frac{(i-1)t}{2^n}\right)\right)^4.$$

Show that this limit is zero almost surely.

Exercise 2. Let $(M_t,\, t \in \mathbb{R}_+)$ be a continuous square-integrable martingale with independent increments. Show that $\langle M\rangle_t = \mathbb{E}(M_t^2) - \mathbb{E}(M_0^2)$ a.s., for all $t \in \mathbb{R}_+$ (so $(\langle M\rangle_t,\, t \in \mathbb{R}_+)$ is a deterministic process in this case).

Exercise 3. a) Let $(M_t,\, t \in \mathbb{R}_+)$ be a continuous martingale, which is moreover *increasing*, that is, $M_s \leq M_t$ a.s. for all $t > s \geq 0$. Show that $M_t = M_0$ a.s., for all $t \in \mathbb{R}_+$.

Remark: By the way, this fact ensures that the Doob decomposition of a submartingale is unique.

b) Let $(M_t,\, t \in \mathbb{R}_+)$ be a continuous square-integrable martingale such that $\langle M\rangle_t = 0$ for all $t \in \mathbb{R}_+$. Show that $M_t = M_0$ a.s., for all $t \in \mathbb{R}_+$.