MIDTERM
Monday November 12, 2007, 13:15-17:00
This exam has 5 problems and 80 points in total.

Instructions

- You are allowed to use 1 sheet of paper for reference. No mobile phones or calculators are allowed in the exam.

- You can attempt the problems in any order as long as it is clear which problem is being attempted and which solution to the problem you want us to grade.

- If you are stuck in any part of a problem do not dwell on it, try to move on and attempt it later.

- Please solve every problem on separate paper sheets.

- It is your responsibility to number the pages of your solutions and write on the first sheet the total number of pages submitted.

Good Luck!
**Problem 1**

**[12 pts]**

[3 pts] (a) Find the DTFT of \( x[n] = \frac{1}{2}2^{n+1}u[-n] + j3^{-n}u[n] \).

[5 pts] (b) Given the same \( x[n] \) whose DTFT \( X(e^{j\omega}) \) we have found in (a), consider the sequence \( y[n] = x^*[-n] + x[n] \). Find its DTFT \( Y(e^{j\omega}) \).

[4 pts] (c) Given the \( x[n] \) whose DTFT \( X(e^{j\omega}) \) we have found in (a), consider the sequence \( z[n] = x^*[1 - n] + x[n - 1] \). Find its DTFT \( Z(e^{j\omega}) \).

**Problem 2**

**[18 pts]**

Consider the system depicted in Figure 1. The impulse responses of the filters are \( h_1[n] = \delta[n - 1] \), \( h_2[n] = 9\delta[n - 1] \), \( h_3[n] = 2\delta[n - 2] \), \( h_4[n] = 3\delta[n - 1] \).

![Figure 1: System with Feedback](image)

[6 pts] (a) Find \( H(z) \), the system function of the overall system, and write its corresponding ROC.

[3 pts] (b) Write down the difference equation corresponding to the overall system.

[6 pts] (c) Compute the impulse response of the overall system, i.e., find \( h[n] \) such that \( y[n] = h[n] * x[n] \).

[1 pts] (d) Is the overall system causal? Why?

[2 pts] (e) Is the overall system BIBO stable? Prove or disprove.
Problem 3

[12 pts]

Assume that \( \tilde{x}[n] \) and \( \tilde{y}[n] \) are two periodic sequences with the same period \( N \). Let \( \tilde{X}[k] \) and \( \tilde{Y}[k] \) be their respective DFS, i.e.,

\[
\tilde{X}[k] = \sum_{n=0}^{N-1} \tilde{x}[n]e^{-j\frac{2\pi}{N}kn}, \quad k = 0, \ldots, N - 1
\]
\[
\tilde{Y}[k] = \sum_{n=0}^{N-1} \tilde{y}[n]e^{-j\frac{2\pi}{N}kn}, \quad k = 0, \ldots, N - 1.
\]

(a) Show that

\[
\sum_{n=0}^{N-1} \tilde{x}^*[n]\tilde{y}[n] = \frac{1}{N} \sum_{k=0}^{N-1} \tilde{X}^*[k]\tilde{Y}[k].
\]

(b) Suppose,

\[
\tilde{x}[n] = 1 + \cos\left(\frac{\pi}{4}n\right)
\]
\[
\tilde{y}[n] = \sin\left(\frac{\pi}{4}n + \frac{\pi}{4}\right).
\]

Find the period \( N \) for these sequences and compute

\[
\sum_{n=0}^{N-1} \tilde{x}^*[n]\tilde{y}[n].
\]
Problem 4

[20 pts]

Consider two sequences \( x[n] \) and \( y[n] \) that have finite-length supports \( N \) and \( M \), respectively. More precisely, \( x[n] = 0 \) for \( n < 0 \) and \( n \geq N \), and \( y[n] = 0 \) for \( n < 0 \) and \( n \geq M \). We suppose that \( x[n] \) and \( y[n] \) are distinct sequences, i.e., there exists an \( n \) for which \( x[n] \neq y[n] \). Assume that \( M > N \).

[4 pts] (a) Let us take the \( M \)-point DFTs, which sample the corresponding DTFTs at \( M \) uniformly-spaced points, i.e.,

\[
X_1[k] = \sum_{n=0}^{M-1} x[n] e^{-j \frac{2\pi}{M} kn}, \quad k = 0, \ldots, M - 1
\]

\[
Y_1[k] = \sum_{n=0}^{M-1} y[n] e^{-j \frac{2\pi}{M} kn}, \quad k = 0, \ldots, M - 1.
\]

Suppose we claim that \( X_1[k] = Y_1[k] \), \( k = 0, \ldots, M - 1 \). If this can be true, give an example. If not, give a proof.

[8 pts] (b) Now, let us sample the corresponding DTFTs at \( N \) uniformly spaced points, i.e.,

\[
X_2[k] = \sum_{n=0}^{M-1} x[n] e^{-j \frac{2\pi}{N} kn}, \quad k = 0, \ldots, N - 1
\]

\[
Y_2[k] = \sum_{n=0}^{M-1} y[n] e^{-j \frac{2\pi}{N} kn}, \quad k = 0, \ldots, N - 1.
\]

Again, suppose we claim that \( X_2[k] = Y_2[k] \), \( k = 0, \ldots, N - 1 \). If this can be true give an example. If not, give a proof.

[8 pts] (c) Assume that \( M = 2N \), and that

\[
y[n] = \begin{cases} 
  x[n], & n = 0, \ldots, N - 1 \\
  x[n - N], & n = N, \ldots, M - 1 
\end{cases}
\]

and consider the sequence

\[
z[n] = y[n] - x[n].
\]

Given that you know that \( z[n] \) has been generated as in (1)–(2), we want to find the smallest number of samples of the DTFT of \( z[n] \) that would be sufficient to reconstruct \( z[n] \). More precisely, suppose the DTFT of \( z[n] \), given by

\[
Z(e^{j\omega}) = \sum_{n} z[n] e^{-j\omega n},
\]

needs to be sampled as,

\[
Z[k] = Z(e^{j\omega})|_{\omega = \frac{2\pi k}{Q}}, \quad k = 0, \ldots, Q - 1.
\]

We want to ensure that \( z[n] \) is recoverable from the samples \( Z[k], k = 0, \ldots, Q - 1 \). Find the smallest \( Q \) that would be sufficient, given that we know the structure of \( z[n] \) as given in (1)–(2). Also, write down the explicit formula of how we can reconstruct \( z[n] \), given \( Z[k], k = 0, \ldots, Q - 1 \), for this smallest value of \( Q \).
Problem 5

18 pts

Let \( H(z) = \frac{8z^{-3} + 20z^{-2} - 10z^{-1} + 3}{8z^{-2} - 4z^{-1} + 1} \) be the system function of an LTI system.

(a) Is there a causal system with system function \( H(z) \)? If yes, give \( h[n] \) and say whether the system is stable. If no, show why.

(b) Is there an anticausal system with system function \( H(z) \)? If yes, give \( h[n] \) and say whether the system is stable. If no, show why.

(c) Consider the composition of two systems depicted in Figure 2, and assume that when \( x[n] \) is the input to the system, then the output is \( y[n] = p[n] \ast x[n] = 8x[n - 3] + 20x[n - 2] - 10x[n - 1] + 3x[n] \). Is the system causal?

(d) Find the system function \( G(z) \) such that \( p[n] = h[n] \ast g[n] \).

(e) Find the corresponding \( g[n] \). Is the system corresponding to \( g[n] \) stable?

\[
\begin{array}{c}
x[n] \xrightarrow{h[n]} \xrightarrow{g[n]} y[n] \\
p[n] 
\end{array}
\]

Figure 2: Composition of two Systems