PROBLEM 1 (PROBLEM 5.6 IN THE BOOK).

- $y[n] = x[-n]$

  Linearity: $\mathcal{H}\{ax_1[n]+bx_2[n]\} = ax_1[-n]+bx_2[-n] = a\mathcal{H}\{x_1[n]\}+b\mathcal{H}\{x_2[n]\}$. Therefore, $\mathcal{H}$ is linear.

  Time-invariance: $\mathcal{H}\{x[n-n_0]\} = x[-n-n_0] \neq y[n-n_0]$. Therefore, $\mathcal{H}$ is not time-invariant.

  Stability: If $|x[n]| \leq M$, then $|\mathcal{H}\{x[n]\}| \leq M$. Therefore, $\mathcal{H}$ is BIBO stable.

  Causality: For negative time indices, the output depends on the future values of the input. Therefore, $\mathcal{H}$ is not causal.

- $y[n] = e^{j\omega n}x[n]$

  Linearity: $\mathcal{H}\{ax_1[n]+bx_2[n]\} = e^{j\omega n}(ax_1[n]+bx_2[n]) = a\mathcal{H}\{x_1[n]\}+b\mathcal{H}\{x_2[n]\}$. Therefore, $\mathcal{H}$ is linear.

  Time-invariance: $\mathcal{H}\{x[n-n_0]\} = e^{j\omega n}x[n-n_0] = e^{j\omega n_0}y[n-n_0]$. Therefore, $\mathcal{H}$ is not time-invariant (unless $\omega = 0$).

  Stability: If $|x[n]| \leq M$, then $|\mathcal{H}\{x[n]\}| \leq M$. Therefore, $\mathcal{H}$ is BIBO stable.

  Causality: The output at any given time depends only on the current input. Therefore, $\mathcal{H}$ is causal.

- $y[n] = \sum_{k=n-n_0}^{n+n_0} x[k]$.

  Linearity: $\mathcal{H}\{ax_1[n]+bx_2[n]\} = \sum_{k=n-n_0}^{n+n_0}(ax_1[k]+bx_2[k]) = a\mathcal{H}\{x_1[n]\}+b\mathcal{H}\{x_2[n]\}$. Therefore, $\mathcal{H}$ is linear.

  Time-invariance: $\mathcal{H}\{x[n-n_0]\} = \sum_{k=n-n_0}^{n+n_0} x[k-n_0] = \sum_{k=n-2n_0}^{n} x[k] = y[n-n_0]$. Therefore, $\mathcal{H}$ is time-invariant.

  Stability: If $|x[n]| \leq M$, then $|\mathcal{H}\{x[n]\}| \leq |2n_0+1|M$. Therefore, $\mathcal{H}$ is BIBO stable.

  Causality: $\mathcal{H}$ is not causal.

  Impulse response: For $x[n] = \delta[n], y[n] = h[n]$:

$$h[n] = \begin{cases} 1 & \text{if } |n| \leq |n_0|, \\ 0 & \text{otherwise.} \end{cases}$$

- $y[n] = ny[n-1]+x[n]$, such that if $x[n] = 0$ for $n < n_0$, then $y[n] = 0$ for $n < n_0$. Since $\mathcal{H}$ is recursive, we can not use the same technique as before. Note that all inputs $x[n]$ can be expressed as a linear combination of delayed impulses: $x[n] = \sum_{k=-\infty}^{\infty} x[k]\delta[n-k]$. Therefore, to show that $\mathcal{H}$ is linear or time-invariant, we can restrict the input to delayed impulses. If $x[n] = \delta[n]$, we can obtain $y[n]$ by recursion:

$$h[n] = y[n] = n!u[n] \tag{1}$$

  If $x[n] = a\delta[n]+b\delta[n]$:

$$y[n] = (a+b)n!u[n]$$

  Therefore, $\mathcal{H}$ is linear.

  To check if $\mathcal{H}$ is time-invariant, consider $x[n] = \delta[n-1]$. It is easy to check that $H\delta[n-1] \neq h[n-1]$.

Stability: It is clear from (1) that the system is unstable.

Causality: $\mathcal{H}$ is causal.

PROBLEM 2. For a low pass filter with cutoff frequency ω_b, we have $h[n] = \frac{\sin(\omega_b n)}{\pi n}$. Thus by the modulation theorem the DTFT of $2h[n]\cos(\omega_0 n)$ would be $H_{bp}(e^{j\omega})$.

PROBLEM 3.

- $x[n] = \delta[n-3] + \delta[n+3]$

$$X(z) = \sum x[n]z^{-n} = z^{-3} + z^3$$

There are 3 poles at $z = 0$ and 3 poles at $z = \infty$, so ROC is all z except 0 and ∞.

- $x[n] = na^n u[n]$. From Z-Transform properties,

$$\begin{aligned} & x[n] \leftrightarrow X(z) \quad \text{with ROC} = \text{R} \\ \Rightarrow \quad & nx[n] \leftrightarrow -z\frac{\partial X}{\partial z} \quad \text{with ROC} = \text{R} \end{aligned}$$

Then, let us define $x_1[n] = a^n u[n]$.

$$X_1(z) = \sum_{n=0}^{\infty} (az^{-1})^n$$

This series converges to $\frac{1}{1-az^{-1}}$ if $|az^{-1}| < 1$. Hence its ROC is: $|z| > a$. The Z-transform of $x[n]$ is

$$\begin{aligned} X(z) &= -z\frac{\partial X_1}{\partial z} \\ &= \frac{az^{-1}}{(1-az^{-1})^2} \quad \text{ROC} = \{|z| > a\} \end{aligned}$$

- $x[n] = -na^n u[-n-1]$ This problem is similar to the previous one. Define $x_1 = -a^n u[-n-1]$. Then $X_1(z)$ is computed as:

$$X_1(z) = \sum_{n=-\infty}^{-1} -(az^{-1})^n = \sum_{n=1}^{\infty} -(za^{-1})^n$$

This series converges to $\frac{1}{1-az^{-1}}$ if $|za^{-1}| < 1$, hence ROC$= \{|z| < a\}$. The Z-transform of $x[n]$:

$$\begin{aligned} X(z) &= -z\frac{\partial X_1}{\partial z} \\ &= \frac{az^{-1}}{(1-az^{-1})^2} \quad \text{ROC}=\{|z| < a\} \end{aligned}$$

- $x[n] = 2^n u[n] - 3^n u[-n-1]$

$$x[n] = x_1[n] + x_2[n]$$

From previous results we know that

$$X_1(z) = \frac{2z^{-1}}{(1-2z^{-1})^2} \quad \text{and} \quad X_2(z) = \frac{3z^{-1}}{(1-3z^{-1})^2}$$

where $\text{ROC}_1 = \{|z| > 2\}$ and $\text{ROC}_2 = \{|z| < 3\}$. So the Z-transform of $x[n]$ is $X(z) = X_1(z) + X_(z)$ with ROC=$\text{ROC}_1 \cap \text{ROC}_2 = \{2 < |z| < 3\}$.

- $x[n] = e^{n^4}[\cos(\frac{\pi}{12}n)]u[n] - e^{n^4}[\cos(\frac{\pi}{12}n)]u[n-1]$

$$\begin{aligned} x[n] &= e^{n^4}[\cos(\frac{\pi}{12}n)](u[n] - u[n-1]) \\ &= e^{n^4}[\cos(\frac{\pi}{12}n)]\delta[n] \\ &= e^{0^4}[\cos(\frac{\pi}{12}0)]\delta[n] = \delta[n] \\ \Rightarrow & X(z) = 1 \Longrightarrow \text{ROC= all } z. \end{aligned}$$

PROBLEM 4. 1.

$$H(z) = \frac{Y(z)}{X(z)} = \frac{1 - 0.6z^{-1} - 2.35z^{-2} - 0.9z^{-3}}{1 - z^{-1} + 0.49z^{-2} - 0.49z^{-3}}$$

Taking inverse z-transforms yields:

$$\begin{aligned} & x[n] - 0.6x[n-1] - 2.35x[n-2] - 0.9x[n-3] \\ & \qquad = y[n] - y[n-1] + 0.49y[n-2] - 0.49y[n-3]. \quad (2) \end{aligned}$$

2. Poles of the system are: $z = \pm j0.7, 1$, zeros of the system are: $z = -0.5, -0.9, 2$. Since the system is causal, therefore right sided, ROC would be the region outside the outermost pole. Thus ROC is $|z| > 1$.
3. Since ROC does not contain the unit circle, the system is not stable. Also ROC contains complex numbers with absolute value greater that 1. This implies that the impulse response must tend to zero as $n \to \infty$. For the last statement to be true, a little thought shows that ROC of the inverse system should be $0.9 < |z| < 2$ (due to stability). But this implies that the system is two-sided contradicting the fact that it is causal, so the statement is false.

PROBLEM 5. (i). Find the system response $H(z) = Y(z)/X(z)$, and plot its pole-zero diagram.

$$\begin{aligned} Y(z) &= Y(z)(\frac{9}{2}z^{-1} + \frac{5}{2}z^{-2}) + X(z)(z^{-1} + 1) \\ Y(z)(1 - \frac{9}{2}z^{-1} - \frac{5}{2}z^{-2}) &= X(z)(z^{-1} + 1) \\ \Rightarrow Y(z)/X(z) &= \frac{1 + z^{-1}}{1 - \frac{9}{2}z^{-1} - \frac{5}{2}z^{-2}} \\ Y(z)/X(z) &= \frac{1 + z^{-1}}{(1 + \frac{1}{2}z^{-1})(1 - 5z^{-1})} \\ Y(z)/X(z) &= \frac{12/11}{(1 - 5z^{-1})} + \frac{-1/11}{(1 + \frac{1}{2}z^{-1})} \\ Y(z)/X(z) &= H_1(z) + H_2(z) \end{aligned}$$

There are two poles at $z = 2$ and $z = -1/2$ and one zero at $z = -1$.

(ii). There are 3 different regions of convergence of $H(z)$, $ROC(H(z)) = ROC(H_1(z)) \cap ROC(H_2(z))$ when $|z| < 1/2$, $1/2 < |z| < 5$ and $|z| > 5$

- anticausal, unstable.
  For the system to be anticausal the ROC should point inwards and for the system

to be unstable the ROC should not contain the unit circle. Then, ROC= $\{|z| < 1/2\}$. This means that the ROC of both H_1 and H_2 should point inwards(i.e., both h_1 and h_2 are anticausal).Therefore,

$$h[n] = -\frac{12}{11}(5)^n u[-n-1] + \frac{1}{11}(-\frac{1}{2})^n u[-n-1]$$

- causal, unstable.
  For the system to be causal the ROC should point outwards and for the system to be unstable the ROC should not contain the unit circle. Then the ROC= $\{|z| > 5\}$. This means that the ROC of both H_1 and H_2 should point outwards(i.e., both h_1 and h_2 are causal).Therefore,

$$h[n] = \frac{12}{11}(5)^n u[n] - \frac{1}{11}(-\frac{1}{2})^n u[n]$$

- causal, stable.
  The system can not be both causal and stable, because the region where the system is defined as causal does not include the unit circle.
- noncausal, stable.
  The sytem is stable and noncausal when $ROC = \{1/2 < |z| < 5\}$. This means that h_1 is anticausal h_2 is causal. Therefore,

$$h[n] = -\frac{12}{11}(5)^n u[-n-1] - \frac{1}{11}(-\frac{1}{2})^n u[n].$$