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Handout 8 Signal Processing for Communications
Solutions to Homework 1 April 5, 2009

PROBLEM 1. A discrete signal $x[n]$ has period N if $x[n] = x[n+kN]$ for any integer k.

1. Suppose that $x[n]$ has period N. That is,

$$e^{j\frac{n}{\pi}} = x[n] = x[n+kN] = e^{j\frac{n+kN}{\pi}}.$$

   Which implies $e^{j\frac{kN}{\pi}} = 1$, which cannot be satisfied any integer N. Hence, $x[n]$ is aperiodic.

2. $x[n] = 2+\sin(4\pi n)+2\cos(3\pi n)$. Clearly the constant 2 has no effect on the periodicity of the signal. Since the rest of the signal is the sum of two periodic signals, $x[n]$ is also periodic. Note that $\sin(4\pi n)$ has period 1 (i.e., it is constant), and $\cos(3\pi n)$ has period 4. Therefore $x[n]$ is of period 4.

3. $x[n] = 2\sin(5\pi n)+3\sin(\sqrt{5}\pi n)$. Notice the that $x[n]$ is the sum of a periodic sequence $(2\sin(5\pi n))$ and an aperiodic sequence $(3\sin(\sqrt{5}\pi n))$, therefore is aperiodic.

4. $x[n] = \cos(2\pi n/7)\sin(2\pi n/5)$. We have a product of two periodic signals, with periods 7 and 5. Therefore the period of $x[n]$ is the least common multiple of the periods of the two factors. In this particular case, the period is 35.

PROBLEM 2. (i). For stability we need to check

$$\sum_n |h[n]| < \infty$$

and for causality we need to check

$$h[n] = 0 \text{ for } n < 0.$$

1. $\sum_n |-e^{|2n|}| = 2\sum_{n=1}^{\infty} e^{2n} + 1 = \infty$. (not stable, not causal).
2. $\sum_n |e^{2n}u[-n+1]| = \sum_{n=-\infty}^{1} e^{2n} = \sum_{n=-1}^{\infty} e^{-2n} = e^2 + \frac{1}{1-e^{-2}} < \infty$. (stable, not causal).
3. $\sum_n |(-1)^n u[3n]| = \sum_{n=0}^{\infty} 1 = \infty$. (not stable, causal).
4. $\sum_n |\frac{1}{3^n}u[n] + 4^n u[-n-2]| = \sum_{n=0}^{\infty} \frac{1}{3^n} + \sum_{n=2}^{\infty} \frac{1}{4^n} < \infty$. (stable, not causal).
5. $\sum_n |\frac{1}{(n-1)^2}u[n-2]| = \sum_{n=2}^{\infty} \frac{1}{(n-1)^2} < \infty$. (stable, causal).

(ii). With the input $x[n] = u[n-2] - u[n-4] = \delta[n-2] + \delta[n-3]$, the output is

$$\begin{aligned} y[n] &= x[n] * h[n] = \sum_m x[m]h[n-m] \\ &= \sum_m (\delta[m-2] + \delta[m-3])h[n-m] \\ &= h[n-2] + h[n-3]. \end{aligned}$$

For the stable systems above (2,4,5) the output becomes;

2. $y[n] = e^{2n-4}u[-n+3] + e^{2n-6}u[-n+4]$.

4. $y[n] = \frac{1}{3^{n-2}}u[n-2] + 4^{n-2}u[-n] + \frac{1}{3^{n-3}}u[n-3] + 4^{n-3}u[-n+1]$.

5. $y[n] = \frac{1}{(n-3)^2}u[n-4] + \frac{1}{(n-4)^2}u[n-5]$.

(iii). We have $x[n] = \delta[n] + \delta[n-1] - \delta[n-2] - \delta[n-3]$. The input is the periodized version of $x[n]$. Then the overall response of the system is

$$\begin{aligned} y[n] &= w[n] * h[n] = \sum_m w[m]h[n-m] \\ &= \sum_m \sum_k x[m-4k]h[n-m] \\ &= \sum_m \sum_k x[m-4k]h[n-m] \\ &= \sum_k h[n-4k] + h[n-1-4k] - h[n-2-4k] - h[n-3-4k]. \end{aligned}$$

which is the periodized version of $h[n] + h[n-1] - h[n-2] - h[n-3]$. For the stable systems above

2.

$$y[n] = \sum_k e^{2n}u[-n+1] + e^{2n-2}u[-n+2+4k] - e^{2n-4}u[-n+3+4k] - e^{2n-6}u[-n+4+4k].$$

4.

$$y[n] = \sum_k \frac{1}{3^n}u[n] + 4^n u[-n-2] + \frac{1}{3^{n-1}}u[n-1] + 4^{n-1}u[-n-1] - \frac{1}{3^{n-2}}u[n-2] - 4^{n-2}u[-n] - \frac{1}{3^{n-3}}u[n-3] - 4^{n-3}u[-n+1].$$

5. $y[n] = \sum_k \frac{1}{(n-1)^2}u[n-2] + \frac{1}{(n-2)^2}u[n-3] - \frac{1}{(n-3)^2}u[n-4] - \frac{1}{(n-4)^2}u[n-5]$.

PROBLEM 3. 1.

$$\begin{aligned} \sum_{n=1}^{\infty} \frac{1}{2^n} + j\frac{1}{3^n} &= \sum_{n=1}^{\infty} \frac{1}{2^n} + j\sum_{n=1}^{\infty} \frac{1}{3^n} \\ &= \frac{1/2}{1-1/2} + j\frac{1/3}{1-1/3} = 1 + \frac{j}{2} \end{aligned}$$

2. $\sum_{n=1}^{\infty} (\frac{j}{3})^n = \frac{j/3}{1-j/3} = -\frac{1}{10} + j\frac{3}{10}$

3. Let $z = a + jb$. We have

$$z^{-1} = z^* = \frac{a-jb}{a^2+b^2} = a - jb \Longrightarrow a^2 + b^2 = 1.$$

I.e., the set satisfying $z^{-1} = z^*$ consists of all the complex numbers on the unit circle.

4. $z_i^3 = e^{j2\pi} \Longrightarrow z_i = e^{j\frac{2\pi}{3}i} \quad z \in \{1, -\frac{1}{2} + j\frac{\sqrt{3}}{2}, -\frac{1}{2} - j\frac{\sqrt{3}}{2}\}$

5. $\prod_{n=1}^{\infty} e^{j\pi/2^n} = e^{j\pi \sum_{n=1}^{\infty} 1/2^n} = e^{j\pi} = -1$

PROBLEM 4. The fact that any vector $\mathbf{z}$ can be represented in the basis $\{\mathbf{x}^{(k)}\}_{k=0,\ldots,N-1}$ follows by the definition of a basis. Now suppose that $\mathbf{z}$ has two distinct representations $\{\alpha_k\}_{k=0,\ldots,N-1} \neq \{\beta_k\}_{k=0,\ldots,N-1}$. That is,

$$\mathbf{z} = \sum_{k=0}^{N-1} \alpha_k \mathbf{x}^{(k)}, \qquad \mathbf{z} = \sum_{k=0}^{N-1} \beta_k \mathbf{x}^{(k)}.$$

We can then write

$$\mathbf{0} = \mathbf{z} - \mathbf{z} = \sum_{k=0}^{N-1} (\alpha_k - \beta_k)\mathbf{x}^{(k)} \neq 0,$$

a contradiction. Therefore, $\mathbf{z}$ is uniquely represented in the basis $\{\mathbf{x}^{(k)}\}_{k=0,\ldots,N-1}$.

PROBLEM 5.

$$\begin{aligned}
X[k] &= \sum_{n=0}^{N-1} \cos\left((2\pi/N)Ln + \phi\right) e^{-j2\pi kn/N} \\
&= \sum_{n=0}^{N-1} \frac{1}{2}\left(e^{j(2\pi/N)Ln+\phi} + e^{-j(2\pi/N)Ln-\phi}\right) e^{-j2\pi kn/N} \\
&= \sum_{n=0}^{N-1} \frac{1}{2}\left(e^{-j(2\pi/N)n(k-L)} e^{j\phi} + e^{-j(2\pi/N)n(k+L)} e^{-j\phi}\right).
\end{aligned}$$

Note that $\frac{1}{N}\sum_{n=0}^{N-1} e^{-j(2\pi/N)n(k-L)}$ is non-zero and equal to 1 only when $k = L + mN$ where $m \in \mathbb{Z}$. Hence,

$$\sum_{n=0}^{N-1} \frac{1}{2}\left(e^{-j(2\pi/N)n(k-L)} e^{j\phi} + e^{-j(2\pi/N)n(k+L)} e^{-j\phi}\right) = \frac{N}{2}\left(\delta[k-L]e^{j\phi} + \delta[k+L]e^{-j\phi}\right)$$

PROBLEM 6.

$$X[k] = \sum_{n=0}^{3} x[n] e^{-2\pi kn/4} = a + (-j)^k b + (-1)^k c + j^k d.$$

$$\begin{aligned}
X[0] &= a + b + c + d \\
X[1] &= a - jb - c + jd \\
X[2] &= a - b + c - d \\
X[3] &= a + jb - c - jd
\end{aligned}$$

For DFT to be real we need $X[k] = X^*[k]$. For this to hold we need {a,b,c,d} to be real and also $b = d$.