Handout 23 Signal Processing for Communications
Solutions to Homework 11 June 4, 2009

PROBLEM 1.

1.

$$\begin{aligned} &\int x(t)e^{-j\Omega t}dt = X(j\Omega) \\ \Rightarrow &\int x(t-\tau)e^{-j\Omega t}dt = e^{-j\Omega\tau}\int x(\tilde{t})e^{-j\Omega\tilde{t}}d\tilde{t} \\ = &\quad e^{-j\Omega\tau}X(j\Omega) \end{aligned}$$

2.

$$\begin{aligned} &\int x(\tau)e^{-jt\tau}d\tau = X(jt) \\ &\int X(jt)e^{-j\Omega t}dt = \int\int x(\tau)e^{-jt\tau}e^{-j\Omega t}dtd\tau \\ = &\int x(\tau)\int e^{-jt(\tau+\Omega)}dtd\tau = 2\pi\int x(\tau)\delta(\tau+\Omega)d\tau \\ = &\quad 2\pi x(-\Omega) \end{aligned}$$

3.

$$\begin{aligned} &\int x(at)e^{-j\Omega t}dt = \frac{1}{a}\int x(\tilde{t})e^{-j\Omega\tilde{t}/a}d\tilde{t} \\ = &\quad \frac{1}{a}X(j\frac{\Omega}{a}) \end{aligned}$$

PROBLEM 2.

1.

$$\begin{aligned} &y(t) = \int x_1(\tau)x_2(t-\tau)d\tau \\ \Rightarrow &\int y(t)e^{-j\Omega t}dt = \int\int x_1(\tau)x_2(t-\tau)e^{-j\Omega t}d\tau dt \\ = &\int x_1(\tau)\int x_2(t-\tau)e^{-j\Omega t}dtd\tau \\ = &\int x_1(\tau)X_2(j\Omega)e^{-j\Omega\tau}d\tau = X_2(j\Omega)\int x_1(\tau)e^{-j\Omega\tau}d\tau \\ = &\quad X_1(j\Omega)X_2(j\Omega) \end{aligned}$$

2.

$$\begin{aligned}
&\int |x_1(t)|^2 dt = \int x_1(t)x_1^*(t)dt \\
=&\frac{1}{(2\pi)^2}\int\int\int X_1(j\Omega)X_1^*(j\tilde{\Omega})e^{j\Omega t}e^{-j\tilde{\Omega}t}d\Omega d\tilde{\Omega}dt \\
=&\frac{1}{(2\pi)^2}\int\int X_1(j\Omega)X_1^*(j\tilde{\Omega})\int e^{jt(\Omega-\tilde{\Omega})}dtd\Omega d\tilde{\Omega} \\
=&\frac{1}{(2\pi)}\int\int X_1(j\Omega)X_1^*(j\tilde{\Omega})\delta(\Omega-\tilde{\Omega})d\tilde{\Omega}d\Omega \\
=&\frac{1}{(2\pi)}\int X_1(j\Omega)X_1^*(j\Omega)d\Omega = \frac{1}{(2\pi)}\int |X_1(j\Omega)|^2 d\Omega
\end{aligned}$$

PROBLEM 3.

1.

$$\begin{aligned}
& x(t) = \frac{1}{2\pi}\int X(j\Omega)e^{j\Omega t}d\Omega \\
\Rightarrow\ & \frac{dx(t)}{dt} = \frac{1}{2\pi}\int X(j\Omega)\frac{de^{j\Omega t}}{dt}d\Omega \\
=\ & \frac{1}{2\pi}\int j\Omega X(j\Omega)e^{j\Omega t}d\Omega \\
\Rightarrow\ & x'(t) \overset{FT}{\leftrightarrow} j\Omega X(j\Omega)
\end{aligned}$$

2.

$$\begin{aligned}
& H(e^{j\omega}) = \frac{j\omega}{T_s} \\
& h[n] = \begin{cases} \frac{1}{2\pi}\int_{-\pi}^{\pi}\frac{j\omega}{T_s}d\omega & \text{for } n=0 \\ \frac{1}{2\pi}\int_{-\pi}^{\pi}\frac{j\omega}{T_s}e^{j\omega n}d\omega & \text{otherwise} \end{cases} \\
& \Rightarrow \frac{1}{2\pi}\int_{-\pi}^{\pi}\frac{j\omega}{T_s}d\omega = \frac{j}{2\pi T_s}\frac{\omega^2}{2}\Big|_{-\pi}^{\pi} = 0 \\
& \text{and } \frac{1}{2\pi}\int_{-\pi}^{\pi}\frac{j\omega}{T_s}e^{j\omega n}d\omega = \frac{j}{2\pi T_s}\left(\frac{\pi e^{j\pi n}}{jn} + \frac{\pi e^{-j\pi n}}{jn} + \frac{e^{j\omega n}}{n^2}\Big|_{-\pi}^{\pi}\right) = \frac{cos(\pi n)}{T_s n} = \frac{(-1)^n}{T_s n} \\
& h[n] = \begin{cases} 0 & \text{for } n=0 \\ \frac{(-1)^n}{T_s n} & \text{otherwise} \end{cases}
\end{aligned}$$

PROBLEM 4.

1. Bandwidth of the system is $2\Omega_0$. Then minimum sampling period becomes $\frac{\pi}{2\Omega_0}$.
2. Figure 1 shows the DTFT of the signal of the sampled signal $x_a[n]$ with a sampling period π/Ω_0

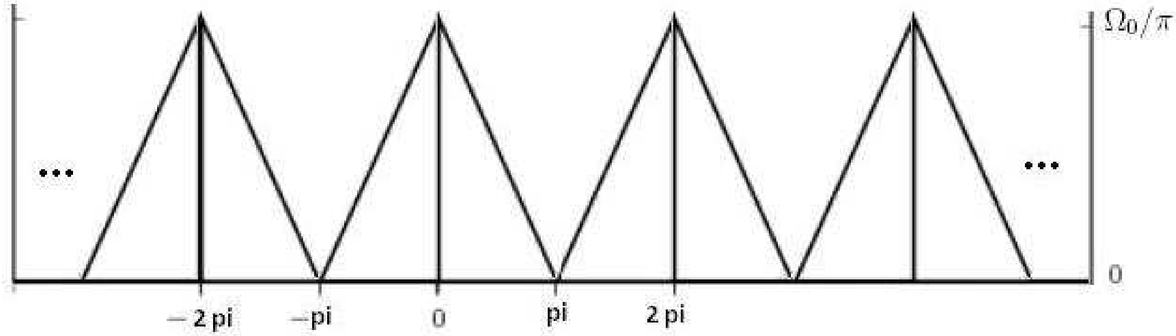

Figure 1: DTFT of discrete signal $x_a[n]$

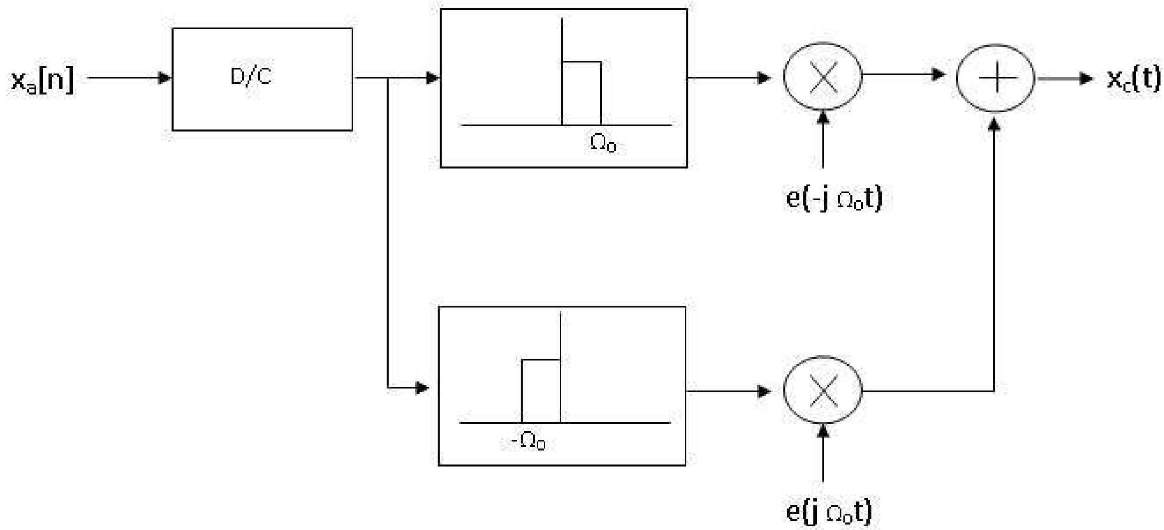

Figure 2: The reconstruction of $x_c(t)$

3. First we convert discrete signal to continuous time. Then we filter the signal in order to get one side lobe, and we shift the corresponding signal(cos modulation). If you repeat the same procedure for the other side lobe, we get the original signal.

4. When we shift in frequency domain in order to prevent aliasing we need to be sure that the baseband $2\Omega_0$ is large enough. That is

$$2(\Omega_1 - \Omega_0) \leq 2\Omega_0$$
$$= \Omega_1 \leq 2\Omega_0$$

   If the baseband($2\Omega_0$) is not large enough or the bandwidth($2(\Omega_1 - \Omega_0)$) is not small enough, then the smallest sampling frequency becomes Nyquist frequency.

$$\Omega_s = \begin{cases} 2(\Omega_1 - \Omega_0), & \text{if } \Omega_1 \leq 2\Omega_0 \\ 2\Omega_1, & \text{else} \end{cases}$$

PROBLEM 5. $Y_c(j\Omega)$ has non zero values between $[1000\pi, 2000\pi]$. That is to say, we are not interested in the values of the input $x(t)$ on the frequencies larger than 2000π and we can sample with periods higher than the Nyquist rate allowing us to have aliasing up to

2000π(Shifts by 6000π). Then the sampling frequency is $\Omega_s = 6000\pi$ and the sampling period is $\frac{1}{3000}$.

$$X(e^{j\omega}) = 3000 \sum_k X(3000\omega - 6000\pi k)$$

$$H(e^{j\omega}) = \begin{cases} 3000|\omega| \text{ or } 3000\omega, & \pi/3 < |\omega| < 2\pi/3 \\ 0, & \text{else} \end{cases}$$

If we assume that $H(e^{j\omega}) = 3000|\omega|$, then

$$\begin{aligned}
h[n] &= \frac{3000}{2\pi} \int_{-\pi}^{\pi} H(e^{j\omega}) e^{j\omega n} d\omega \\
&= \frac{3000}{2\pi} \left(- \int_{-2\pi/3}^{-\pi/3} \omega e^{j\omega n} d\omega + \int_{\pi/3}^{2\pi/3} \omega e^{j\omega n} d\omega \right) \\
&= \frac{3000}{2\pi} \left(- \int_{2\pi/3}^{\pi/3} \omega e^{-j\omega n} d\omega + \int_{\pi/3}^{2\pi/3} \omega e^{j\omega n} d\omega \right) \\
&= \frac{3000}{2\pi} \left(\int_{\pi/3}^{2\pi/3} \omega (e^{-j\omega n} + e^{j\omega n}) d\omega \right) \\
&= \frac{3000}{\pi} \left(\int_{\pi/3}^{2\pi/3} \omega \cos(\omega n) d\omega \right) \\
&= \frac{3000}{\pi} \left(\frac{\omega \sin(\omega n)}{n} \bigg|_{\pi/3}^{2\pi/3} - \frac{1}{n} \int_{\pi/3}^{2\pi/3} \sin(\omega n) d\omega \right) \\
&= \frac{3000}{\pi} \left(\frac{\omega \sin(\omega n)}{n} \bigg|_{\pi/3}^{2\pi/3} + \frac{\cos(\omega n)}{n^2} \bigg|_{\pi/3}^{2\pi/3} \right)
\end{aligned}$$

The magnitude of frequency responses $X(j\Omega)$, $X(e^{j\omega})$ and $H(e^{j\omega})$ are shown in Figure 3

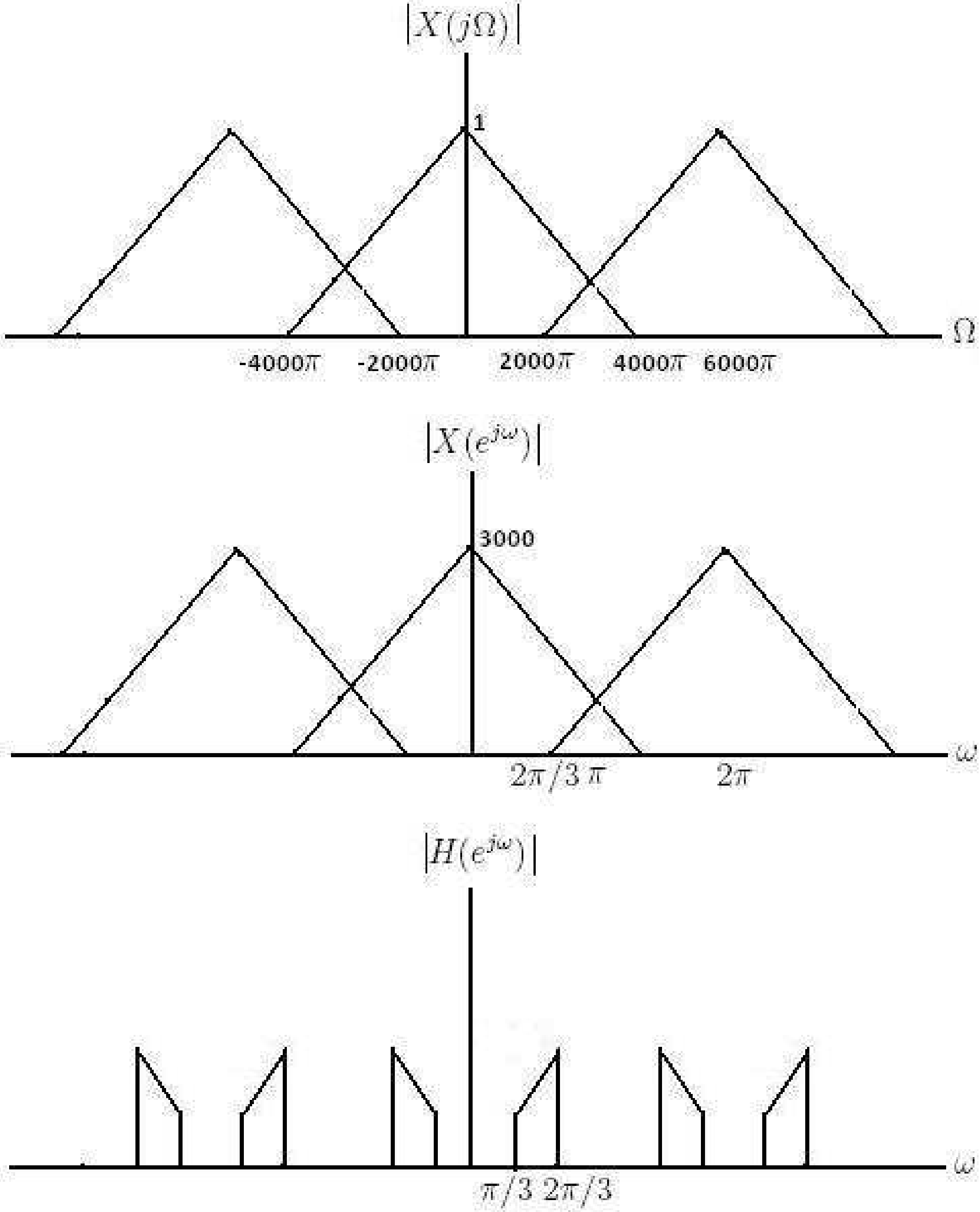

Figure 3: FT of sampled signal $x(nT_s)$ (up), DTFT of the discrete signal $x[n]$ (middle), DTFT of $h[n]$ (bottom)