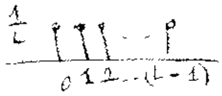
PROBLEM 1:

1) $x[n] \rightarrow \underset{\text{MAF}}{\boxed{h[n]}} - \underset{\text{MAF}}{\boxed{h[n]}} - y[n]$, $y[n] = (x[n] * h[n] * h[n])$

impulse response $\Rightarrow h[n] = \frac{1}{L} \sum_{k=0}^{L-1} \delta[n-k]$



a) linearity: (additivity + homogenity)

$$x_1[n] \longrightarrow \frac{1}{L} \sum_{k=0}^{L-1} x_1[n-k] = y_1[n]$$

$$x_2[n] \longrightarrow \frac{1}{L} \sum_{k=0}^{L-1} x_2[n-k] = y_2[n]$$

$$(a x_1[n] + b x_2[n]) = f[n] \longrightarrow \frac{1}{L} \sum_{k=0}^{L-1} f[n-k] = \frac{1}{L} \sum_{k=0}^{L-1} (a x_1[n-k] + b x_2[n-k])$$

$$= \frac{a}{L} \sum_{k=0}^{L-1} x_1[n-k] + \frac{b}{L} \sum_{k=0}^{L-1} x_2[n-k]$$

$$= a \cdot y_1[n] + b\, y_2[n]$$ ✓

b) time-invariance

$x[n] \rightarrow y[n]$, $\quad y[n] = \frac{1}{L} \sum_{k=0}^{L-1} x[n-k]$

$x[n+m] \xrightarrow{?} y[n+m]$

$$x[n+m] \longrightarrow \frac{1}{L} \sum_{k=0}^{L-1} x[n+m-k] = y[n+m]$$ time-invariant ✓

c) causality. $(h[n] = 0 \quad \forall n < 0)$

$$h[n] = \frac{1}{L} \sum_{k=0}^{L-1} \delta[n-k] = \begin{cases} 0 & n < 0,\ n \geq L \\ \frac{1}{L} & 0 \leq n \leq L \end{cases}$$ ✓ causal.

d) stability:

$$\left(\sum |h[n]| < \infty \right)$$

$$\delta[n-k] = \begin{cases} 1 & n = k \\ 0 & \text{else} \end{cases}$$

$$\sum_{n=-\infty}^{\infty} \left| \frac{1}{L} \sum_{k=0}^{L-1} \delta[n-k] \right| = \frac{1}{L} \sum_{n=-\infty}^{\infty} \sum_{k=0}^{L-1} \delta[n-k]$$

$$= \frac{1}{L} \sum_{n=-\infty}^{\infty} \left(\delta[n] + \delta[n-1] + \dots + \delta[n-L+1] \right)$$

$$= \frac{1}{L} \cdot L \cdot \delta[0] = 1$$ ✓ stable

→ If two LTI systems are cascaded, then the overall system is also LTI. So here it is enough to show that one moving average filter is a stable, linear, time-invariant filter.

1.) e) $h[n] * h[n] = \left(\frac{1}{L}\sum_{k=0}^{L-1}\delta[n-k]\right) * \left(\frac{1}{L}\sum_{\ell=0}^{L-1}\delta[n-\ell]\right)$

$$h_{overall}[n] = \sum_{m=-\infty}^{\infty} h[n-m]\cdot h[m] = \sum_{m=-\infty}^{\infty}\left(\frac{1}{L}\sum_{k=0}^{L-1}\delta[n-m-k]\cdot\frac{1}{L}\sum_{\ell=0}^{L-1}\delta[m-\ell]\right)$$

$$= \frac{1}{L^2}\sum_{m=-\infty}^{\infty}\sum_{k=0}^{L-1}\sum_{\ell=0}^{L-1}\delta[n-m-k]\cdot\delta[m-\ell] \quad , \quad \delta[m-\ell] = \begin{cases}1 & m=\ell\\ 0 & \text{else}\end{cases}$$

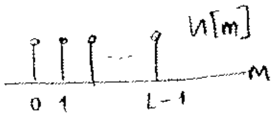
$m = \ell$

$m = n-k$

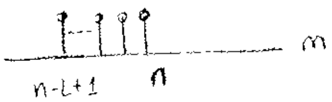
$\Rightarrow \ell = n-k$

$\Rightarrow n = \ell + k \quad \begin{cases}0 \le \ell \le L-1\\ 0 \le k \le L-1\end{cases}$

$0 \le n \le 2L-2 \Rightarrow$ so maximum length of the overall filter is $2L-2+1 = \underline{\underline{2L-1}}$



h[n-m]



for $n < 0 \Rightarrow h[m] * h[n-m] = 0$

for $0 \le n \le L-1 \Rightarrow h[m]*h[n-m] = \frac{1}{L^2}\sum_{\ell=0}^{n} 1 = \frac{n+1}{L^2}$

for $L-1 \le n \le 2L-2 \Rightarrow h[m]*h[n-m] = \frac{1}{L^2}\sum_{\ell=n}^{2L-2} 1 = \frac{2L-2-n+1}{L^2} = \underline{\underline{\frac{2L-n-1}{L^2}}}$

$$\Rightarrow h_{overall}[n] = \begin{cases}\frac{n+1}{L^2} & , 0 \le n \le L-1\\ \frac{2L-n-1}{L^2} & , L-1 \le n \le 2L-2\end{cases}$$ triangle function

2.) Instead of applying $h_{overall}[n]$, apply one moving average of length $2L_0$.

$h_{2L}[n] = \frac{1}{2L}\sum_{k=0}^{2L-1}\delta[n-k]$, $h_{overall}[n] = tri[n]$

$h_{2L}[n] \neq h_{overall}[n]$ {two impulse responses are not the same so these two systems are different.}

$x[n] \rightarrow \boxed{h_{2L}[n]} \rightarrow y_{2L}[n] = \frac{1}{2L}\sum_{k=0}^{2L-1} x[n-k]$

3.) we sample at 8000 Hz, noise at 1000 Hz,

→ the average of a sin(t) function is zero

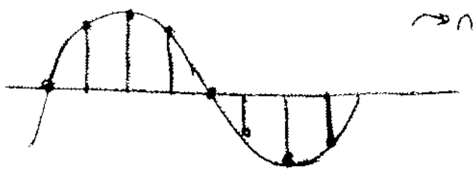
(continuous)

→ we should find L such that we cover N period of sin function. $(N \in \{1,2,3,..\})$

3) for $f_s = 8000$ Hz $\quad T_s = \frac{1}{f_s} = 0,125$ ms

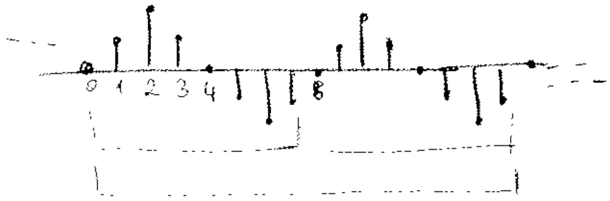
noise, $f_n = 1000$ Hz $\quad T_n = \frac{1}{f_n} = 1$ ms

$$\frac{8000}{1000} = \frac{1\text{ms}}{0,125\text{ms}} = 8 \text{ samples in one period of } \sin(2000\pi t)$$



$\rightarrow n(t)$ {continuous}

$n[k]$ {discrete}



if we take $L = 8N$ samples then we take the average of one sinusoid (N periods, $N \in \{1, 2, \ldots\}$)

PROBLEM 2:

1.) $x(t) = \sin(10\pi t)\cos(30\pi t)\sin(50\pi t) - \sin(5\pi t)\cos(20\pi t)$

$fs = 32\,Hz$

we should represent $x(t)$ in terms of summation of sinusoidals.

$$x(t) = \sin(10\pi t)\cdot\left[\frac{1}{2}\left(\sin(\underbrace{30\pi t + 50\pi t}_{80\pi t}) - \sin(\underbrace{30\pi t - 50\pi t}_{-20\pi t})\right)\right]$$

$$- \frac{1}{2}\left(\sin(\underbrace{5\pi t + 20\pi t}_{25\pi t}) + \sin(\underbrace{5\pi t - 20\pi t}_{-15\pi t})\right)$$

$$= \frac{1}{2}\cdot\frac{1}{2}\left[\cos(10\pi t - 80\pi t) - \cos(10\pi t + 80\pi t) - \cos(10\pi t + 20\pi t) + \cos(10\pi t - 20\pi t)\right]$$

$$- \frac{1}{2}\sin(25\pi t) + \frac{1}{2}\sin(15\pi t)$$

$$= \frac{1}{4}\cos(70\pi t) - \frac{1}{4}\cos(90\pi t) - \frac{1}{4}\cos(30\pi t) + \frac{1}{4}\cos(10\pi t) - \frac{1}{2}\sin(25\pi t) + \frac{1}{2}\sin(15\pi t)$$

with $fs = 32\,Hz$, we can correctly reconstruct signals with $f \leq \frac{fs}{2} = 16\,Hz$

for others we will have $\{f_r = f + m fs\}$ and we need to satisfy

$|f + m fs| \leq 16$, $m \in \{\ldots -1, 0, 1\}$ because of ideal interpolator.

$35 > fs/2$
$45 > fs/2$ $\left.\begin{array}{l}\cos(70\pi t)\\ \cos(90\pi t)\end{array}\right\}$ can't be reconstructed correctly

$|35\,Hz + m fs| \leq 16$, $35 - 32 = \underline{3\,Hz} < 16 \rightarrow \cos(6\pi t)$

$|90 + m fs| \leq 16$, $90 - 3.(32) = \underline{-6\,Hz} \rightarrow \cos(-12\pi t) = \cos(12\pi t)$

so $x_r(t) = \frac{1}{4}\cos(6\pi t) - \frac{1}{4}\cos(12\pi t) - \frac{1}{4}\cos(15\pi t) + \frac{1}{4}\cos(10\pi t) - \frac{1}{2}\sin(25\pi t)$

$+ \frac{1}{2}\sin(5\pi t)$

2.) for smallest sampling frequency, we should select smallest frequency with which we can reconstruct every signal correctly.

$$\left.\begin{array}{l}\cos(70\pi t) \rightarrow fs \geq 2f = 70\\ \cos(90\pi t) \rightarrow fs \geq 2f = 90\\ \cos(30\pi t) \rightarrow fs \geq 2f = 30\\ \cos(10\pi t) \rightarrow fs \geq 10\\ \sin(25\pi t) \rightarrow fs \geq 25\\ \sin(15\pi t) \rightarrow fs \geq 15\end{array}\right\}$$ minimum sampling frequency $\Rightarrow fs = \underline{90\,Hz}$

PROBLEM 3:

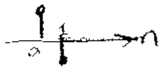
1.) $h_1[n] = \delta[n] - \delta[n-1]$, for all n

$x[n] = 1$, for all n

$$y[n] = x[n] * h_1[n] = \sum_{m=-\infty}^{\infty} \underbrace{x[m]}_{1} h[n-m] = \sum_{m=-\infty}^{\infty} 1 \cdot h[n-m] = \sum_{m=-\infty}^{\infty} \left(\delta[n-m] - \delta[n-m-1]\right)$$

$$y[n] = \sum_{m=-\infty}^{\infty} \delta[n-m] - \sum_{m=-\infty}^{\infty} \delta[n-m-1] = 1 - 1 = \underline{0}$$

$\delta[n-m] = \begin{cases} 1 & m=n \\ 0 & \text{else} \end{cases}$, since summation is over $(-\infty, \infty)$ m can reach any value of n

the system is differentiator 

2.) $x[n] = n$ for all n

$$y[n] = \sum_{m=-\infty}^{\infty} x[m]\, h[n-m] = \sum_{m=-\infty}^{\infty} x[n-m]\, h[m] = \sum_{m=-\infty}^{\infty} (n-m)\left[\delta[m] - \delta[m-1]\right]$$

$$= \sum_{m=-\infty}^{\infty} \underbrace{(n-m)\,\delta[m]}_{m=0} - \sum_{m=-\infty}^{\infty} \underbrace{(n-m)\,\delta[m-1]}_{m=1} = n - (n-1) = n - n + 1 = \underline{1}$$

3.) $$y[n] = x[n] * h_2[n] = \sum_{m=-\infty}^{\infty} x[m]\, h[n-m] = \sum_{m=1}^{n} 1 \cdot 1 = n - 1 + 1 = \underline{n}$$

$x[m] = \begin{cases} 1 & m \geq 1 \\ 0 & \text{else} \end{cases}$

$h[n-m] = \begin{cases} 1 & n-m \geq 0 \\ 0 & \text{else} \end{cases}$ or $m \leq n$

4.) $$y[n] = x[n] * h_2[n] = \sum_{m=-\infty}^{\infty} x[m]\, h[n-m] = \sum_{m=1}^{n} m \cdot 1 = \sum_{m=1}^{n} m = \underline{\frac{n \cdot (n+1)}{2}}$$

$x[m] = \begin{cases} m & m > 0 \\ 0 & \text{else} \end{cases}$

$\rightarrow x[n] \longrightarrow \boxed{h_1[n]} \longrightarrow \boxed{h_2[n]} \longrightarrow y[n]$

$x[n] = n$, $n \geq 0$ $\longrightarrow y[n] = ?$ since the two system h_1, h_2 are linear-time invariant, the overall impulse response is the convolution of h_1 & h_2.

Also, since the system is LTI, we can change the order of the filters $(h_1 * h_2) = (h_2 * h_1)$

$x[n] \longrightarrow \boxed{h_2[n]} \longrightarrow \boxed{h_1[n]} \longrightarrow y[n]$

$$h_{overall}[n] = h_1[n] * h_2[n] = \sum_{m=-\infty}^{\infty} h_1[m]\, h_2[n-m] = \sum_{m=-\infty}^{\infty} h_1[n-m]\, h_2[m]$$

$$= \sum_{m=0}^{\infty} \left(\delta[n-m] - \delta[\underbrace{n-m-1}]\right) \cdot 1 \qquad h_2[m] = 1 \quad m \geq 0$$

$\Downarrow$ $n=m$ $\qquad \Downarrow$ $n = m+1$

$n = m+1, \; n \geq 1$

$n = m, \; n \geq 0$

for $n < 0$, $\sum_{m=0}^{\infty} (\delta[n-m] - \delta[n-m-1]) = 0 - 0 = 0$

for $n = 0$, $\sum_{m=0}^{\infty} \delta[-m] - \delta[-m-1] = 1 - 0 = 1$

$m=0 \checkmark \qquad m=-1 \times$

for $n \geq 1$, $\sum_{m=0}^{\infty} (\delta[n-m] - \delta[n-m-1]) = 1 - 1 = \underline{\underline{0}}$

so $h_{overall}[n] = \begin{cases} 1 & n=0 \\ 0 & \text{else} \end{cases} = \underline{\underline{\delta[n]}}$

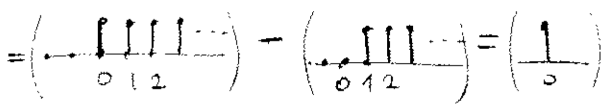
when $x[n] = n, \; n \geq 0 \rightarrow y[n] = x[n] * \delta[n] = \sum_{m=-\infty}^{\infty} x[m] \cdot h[n-m]$

$$= \sum x[m]\, \delta[n-m] = \underline{\underline{x[n]}}$$

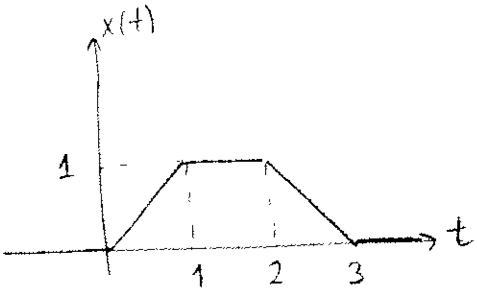
$\Rightarrow$ when we convolve a function with delta function we get the signal itself ($x[n] * \delta[n] = x[n]$), similarly convolving with a shifted delta function gives shifted signal ($x[n] * \delta[n-k] = x[n-k]$)

for example $h_2[n] = 1 \quad n \geq 0 = u[n]$ (step function)

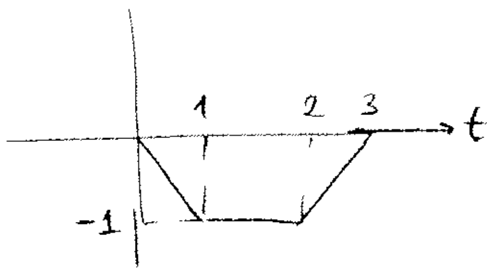
$$h_2[n] * h_1[n] = u[n] * (\delta[n] - \delta[n-1]) = u[n] - u[n-1] = \underline{\underline{\delta[n]}} \text{ easily.}$$



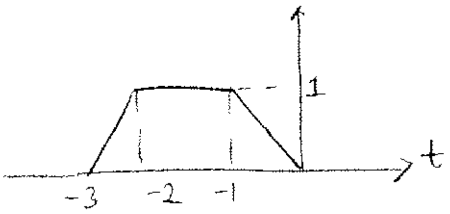
PROBLEM 4:



$-x(t)$

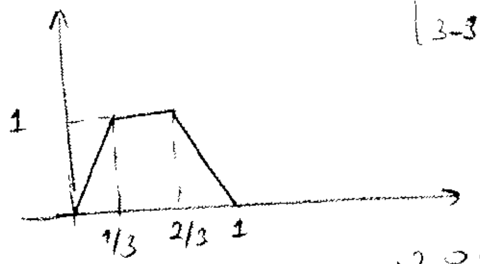


$x(-t)$

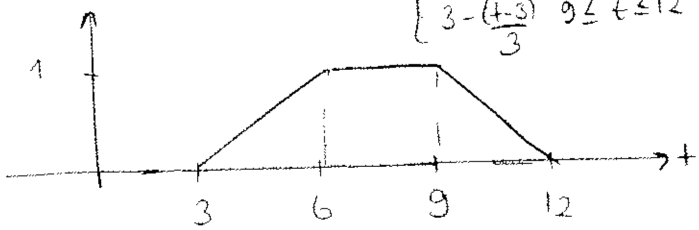


$x(3t)$

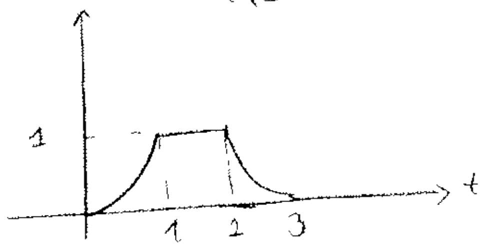
$$x(3t) = \begin{cases} 3t & 0 \le t \le 1/3 \\ 1 & 1/3 \le t \le 2/3 \\ 3-3t & 2/3 \le t \le 1 \end{cases}$$



$$x((t-3)/3) = \begin{cases} \frac{t-3}{3} & 3 \le t \le 6 \\ 1 & 6 \le t \le 9 \\ 3-\frac{(t-3)}{3} & 9 \le t \le 12 \end{cases}$$



$$x^2(t) = \begin{cases} t^2 & 0 \le t \le 1 \\ 1 & 1 \le t \le 2 \\ (3-t)^2 & 2 \le t \le 3 \end{cases}$$



$$x(t^2) = \begin{cases} t^2 & 0 \le |t| \le 1 \\ 1 & 1 \le |t| \le \sqrt{2} \\ 3-t^2 & \sqrt{2} \le |t| \le \sqrt{3} \end{cases}$$