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Handout 10 Introduction to Communication Systems
Solutions to Homework 5 October 23, 2008

PROBLEM 1. 1. $H(P) = 1.8016$ and $H(Q) = 1.9406$.

2. We know that the the discrete entropy is maximized by the uniform distribution. By looking at P and Q we can see that Q is closer to the uniform distribution than P. We can then conclude that $H(Q) > H(P)$.

PROBLEM 2. 1. $\sum_{i=1}^{\infty} p_i = \sum_{i=1}^{\infty} \frac{1}{2^i} = \sum_{i=0}^{\infty} \frac{1}{2^i} - 1 = \frac{1}{1-1/2} - 1 = 2 - 1 = 1$.

2. $\sum_{i=1}^{\infty} p_i \log_2(\frac{1}{p_i}) = \sum_{i=1}^{\infty} \frac{1}{2^i} \log_2(2^i) = \sum_{i=1}^{\infty} \frac{1}{2^i} i$
$= \sum_{i=0}^{\infty} \frac{1}{2^i} i = \frac{1/2}{(1-1/2)^2} = 2$.

PROBLEM 3. 1. Code 1 is not uniquely decodable and not prefix free. For instance if we observe the sequence 010, we do not know if it corresponds to s_2 or $s_3 s_1$. Another way to see that the code is not uniquely decodable is to use the Kraft's inequality. We see that the code does not satisfy Kraft's inequality and hence is not uniquely decodable. Note that Kraft's inequality is a necessary condition and not sufficient. For example consider a code given by $00, 010, 001, 1$. We will not be able to decode 001 uniquely because it can correspnd to s_3 or $s_1 s_4$, but the codeword lengths satisfy Kraft's inequality since $2^{-2} + 2^{-3} + 2^{-3} + 2^{-1} = \frac{1}{4} + \frac{1}{8} + \frac{1}{8} + \frac{1}{2} = 1$.

2. Code 2 is prefix free since no symbol is the prefix of an other one. So it is also uniquely decodable. Also since it is prefix free, it is also instantaneous.

3. Code 3 is not prefix free since s_3 is the prefix of s_4. To see that it is uniquely decodable, assume on the contrary that it is not uniquely decodable. Thus there exists a binary string of length, say n, such that it corresponds to two different symbol streams S_1 and S_2. Let i be the first position where the two symbol streams differ. Since the binary string representing S_1, S_2 is the same, the binary string must be the $1100\ldots0$ from this point onwards. But depending upon whether there are odd number of zeros or even number of zeros, uniquly determines the symbol stream leading to a contradiction that S_1, S_2 differ. Thus the code is uniquely decodable.

PROBLEM 4. 1. We want to prove that $\ln x \leq x - 1$. Taking the exponential on both sides, we want to prove that $x \leq e^{x-1}$. Consider first the case $x \geq 1$. Using Taylor's expansion we have

$$\begin{aligned} e^{x-1} &= 1 + (x-1) + \frac{(x-1)^2}{2!} + \frac{(x-1)^3}{3!} + \ldots \\ &= x + \frac{(x-1)^2}{2!} + \frac{(x-1)^3}{3!} + \ldots \end{aligned}$$

Since $(x-1)^k \geq 0$ for all k and we have

$$\begin{aligned} x &\leq x + \frac{(x-1)^2}{2!} + \frac{(x-1)^3}{3!} + \ldots \\ &= e^{x-1} \end{aligned}$$

Consider the case $0 \leq x < 1$. We will again use the Taylor's expansion for the exponential. But now $x - 1 < 0$. Nevertheless we have that $\frac{(x-1)^{2k}}{(2k)!} + \frac{(x-1)^{2k+1}}{(2k+1)!} \geq 0$ for all k. Thus we have

$$\begin{aligned} x &\leq x + \Big(\frac{(x-1)^2}{2!} + \frac{(x-1)^3}{3!}\Big) + \Big(\frac{(x-1)^4}{4!} + \frac{(x-1)^5}{5!}\Big) + \dots \\ &= e^{x-1} \end{aligned}$$

2. By setting $x = \frac{q_i}{p_i} \geq 0$, we have $\frac{q_i}{p_i} - 1 \geq \ln(\frac{q_i}{p_i}) = \ln(q_i) + \ln(\frac{1}{p_i})$. Thus $\ln(\frac{1}{p_i}) \leq \frac{p_i}{q_i} - 1 - \ln(q_i)$. We can then write

$$\begin{aligned} \sum_{i=1}^{n} p_i \ln\left(\frac{1}{p_i}\right) &\leq \sum_{i=1}^{n} p_i \left(\frac{q_i}{p_i} - 1 - \ln(q_i)\right) \\ &= \sum_{i=1}^{n} q_i - 1 - \sum_{i=1}^{n} p_i \ln(q_i) \\ &= \sum_{i=1}^{n} p_i \ln\left(\frac{1}{q_i}\right) \end{aligned}$$