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Handout 4 Introduction to Communication Systems
Solutions to Homework 2 October 2, 2008

PROBLEM 1. (a) The system is not causal, because the output has a non-zero value at $n = -1$, which means that we get an output even before we have applied the input.

(b) The system is not linear, because if we add $x_1 + x_2$ we get x_3, but $y_3 \neq y_1 + y_2$.

(c) We cannot say whether the system is time-invariant or not from the three test signals.

PROBLEM 2. From the definition of stability we know that if

$$\sum_{m=-\infty}^{\infty} |h(m)| < \infty$$

then the system is stable. So for the first case we know that the sum $\sum_{m=0}^{\infty} \frac{1}{m}$ diverges, hence the system is unstable. Consider the input $x[n] = u[n]$, where

$$u[n] = \begin{cases} 1, & n \geq 0 \\ 0, & \text{else} \end{cases}$$

The output of the system is

$$y[n] = \sum_{m=0}^{n-1} \frac{1}{n-m}$$

Clearly as n increases we cannot bound the output $y[n]$ with a constant even though the input is always 1.

The second system is stable because the sum

$$\sum_{m=0}^{\infty} \frac{1}{m^2} = \frac{\pi^2}{6}$$

converges to a constant. So for any input which is bounded by a constant over all the range of time, the output will also be bounded by a constant.

PROBLEM 3. (a) $x(n)$

(b) $x(n+3)$ is $x(n)$ shifted by 3 to the left

(c) $x(2-n)\delta(n-1) = \begin{cases} x(1) & \text{if } n = 1 \\ 0 & \text{otherwise} \end{cases}$.

(d) $x(n-1)$ is $x(n)$ shifted by 1 to the right and the convolution with $\delta(3-n)$ shifts the signal by 3 to the right. So $x(n-1) \star \delta(3-n)$ is $x(n)$ shifted by 4 to the right.

PROBLEM 4. (a) the output is given by $h(-1) = 1$, $h(0) = 3.25$, $h(1) = 2.5$, $h(2) = 2$, $h(3) = 3.5$, $h(4) = 1.25$
(b) the impulse response is $h(n) = h_1(n) \star h_2(n)$. It is the same if we swap H_1 and H_2 since the convolution is a commutative operator.
This system is also a filter, since we can see that:

- It is linear (we can easily prove that the cascade of two linear systems is also linear)
- It is time invariant (the composition of two invariant systems is also time invariant)
- The domain of the input and output signals is the same (in this case it is $\mathbb{Z}$)

PROBLEM 5. 1. (a) $x[n] = \mathrm{Im}(e^{j\omega n})$

(b) $\mathrm{Im}(\gamma a + \beta b) = \mathrm{Im}(\gamma x + j\gamma y + \beta u + j\beta v)$ where $a = x + jy, b = u + jv$ and $\gamma, \beta \in \mathbb{R}$. Thus

$$\begin{aligned}\mathrm{Im}(\gamma a + \beta b) &= \gamma y + \beta v \\ &= \gamma \mathrm{Im}(a) + \beta \mathrm{Im}(b)\end{aligned}$$

(c) Let the input be $x[n] = e^{j\omega n}$. Convolution gives us

$$\begin{aligned}y[n] &= \sum_{m=-\infty}^{\infty} e^{j\omega(n-m)} h(m) \\ &= \sum_{m=0}^{\infty} e^{j\omega(n-m)} \alpha^m \\ &= e^{j\omega n} \sum_{m=0}^{\infty} (e^{-j\omega}\alpha)^m \\ &= e^{j\omega n} \frac{1}{1 - e^{-j\omega}\alpha}\end{aligned}$$

Thus the amplitude of the output is given by

$$\begin{aligned}\left| \frac{1}{1 - e^{-j\omega}\alpha} \right| &= \left| \frac{1}{1 - \alpha\cos\omega + j\alpha\sin\omega} \right| \\ &= \left| \frac{1 - \alpha\cos\omega - j\alpha\sin\omega}{(1 - \alpha\cos\omega)^2 + (\alpha\sin\omega)^2} \right| \\ &= \frac{1}{\sqrt{(1 - \alpha\cos\omega)^2 + (\alpha\sin\omega)^2}}\end{aligned}$$

Thus the output is given by

$$y[n] = \frac{1}{\sqrt{(1 - \alpha\cos\omega)^2 + (\alpha\sin\omega)^2}} e^{j(\omega n + \phi)}$$

where

$$\tan\phi = \frac{-\alpha\sin\omega}{1 - \alpha\cos\omega}$$

(d) The input signal is $\mathrm{Im}(e^{j\omega n})$. Thus the output is given by

$$\begin{aligned}y[n] &= \sum_{m=-\infty}^{\infty} \mathrm{Im}(e^{j\omega(n-m)}) h(m) \\ &= \mathrm{Im}(\sum_{m=-\infty}^{\infty} e^{j\omega(n-m)} h(m)) \\ &= \mathrm{Im}(\sum_{m=0}^{\infty} e^{j\omega(n-m)} h(m))\end{aligned}$$

The second equality follows from the linearity of the $\mathrm{Im}(\cdot)$ function. Using the solution to the part above we get

$$\begin{aligned} y[n] &= \mathrm{Im}\left(\frac{1}{\sqrt{(1-\alpha\cos\omega)^2+(\alpha\sin\omega)^2}}e^{j(\omega n+\phi)}\right) \\ &= \frac{1}{\sqrt{(1-\alpha\cos\omega)^2+(\alpha\sin\omega)^2}}\sin(\omega n+\phi) \end{aligned}$$

where ϕ is given in the solution to the previous part.

2. The amplitude is given by

$$\frac{1}{\sqrt{(1-\alpha\cos\omega)^2+(\alpha\sin\omega)^2}}$$

For $\omega = 0$ the amplitude is $\frac{1}{1-\alpha}$. For $\omega = \pi$ the amplitude is $\frac{1}{1+\alpha}$.

3. As $\alpha \to 1$ the amplitude at $\omega = 0$ tends to infinity and amplitude at $\omega = \pi$ tends to $\frac{1}{2}$. This is a like a low-pass filter.