Problem 1. (a) The system is not causal, because the output has a non-zero value at $n = -1$, which means that we get an output even before we have applied the input.

(b) The system is not linear, because if we add $x_1 + x_2$ we get $x_3$, but $y_3 \neq y_1 + y_2$.

(c) We cannot say whether the system is time-invariant or not from the three test signals.

Problem 2. From the definition of stability we know that if
\[ \sum_{m=-\infty}^{\infty} |h(m)| < \infty \]
then the system is stable. So for the first case we know that the sum $\sum_{m=0}^{\infty} \frac{1}{m}$ diverges, hence the system is unstable. Consider the input $x[n] = u[n]$, where
\[ u[n] = \begin{cases} 1, & n \geq 0 \\ 0, & \text{else} \end{cases} \]
The output of the system is
\[ y[n] = \sum_{m=0}^{n-1} \frac{1}{n-m} \]
Clearly as $n$ increases we cannot bound the output $y[n]$ with a constant even though the input is always 1.

The second system is stable because the sum
\[ \sum_{m=0}^{\infty} \frac{1}{m^2} = \frac{\pi^2}{6} \]
converges to a constant. So for any input which is bounded by a constant over all the range of time, the output will also be bounded by a constant.

Problem 3. (a) $x(n)$

(b) $x(n+3)$ is $x(n)$ shifted by 3 to the left

(c) $x(2-n)\delta(n-1) = \begin{cases} x(1) & \text{if } n = 1 \\ 0 & \text{otherwise} \end{cases}$

(d) $x(n-1)$ is $x(n)$ shifted by 1 to the right and the convolution with $\delta(3-n)$ shifts the signal by 3 to the right. So $x(n-1) \ast \delta(3-n)$ is $x(n)$ shifted by 4 to the right.

Problem 4. (a) the output is given by $h(-1) = 1$, $h(0) = 3.25$, $h(1) = 2.5$, $h(2) = 2$, $h(3) = 3.5$, $h(4) = 1.25$

(b) the impulse response is $h(n) = h_1(n) \ast h_2(n)$. It is the same if we swap $H_1$ and $H_2$ since the convolution is a commutative operator. This system is also a filter, since we can see that:
- It is linear (we can easily prove that the cascade of two linear systems is also linear)
- It is time invariant (the composition of two invariant systems is also time invariant)
- The domain of the input and output signals is the same (in this case it is \( \mathbb{Z} \))

**Problem 5.**

1. (a) \( x[n] = \text{Im}(e^{j\omega n}) \)

(b) \( \text{Im}(\gamma a + \beta b) = \text{Im}(\gamma x + j\gamma y + \beta u + j\beta v) \) where \( a = x + jy, b = u + jv \) and \( \gamma, \beta \in \mathbb{R} \). Thus

\[
\text{Im}(\gamma a + \beta b) = \gamma y + \beta v = \gamma \text{Im}(a) + \beta \text{Im}(b)
\]

(c) Let the input be \( x[n] = e^{j\omega n} \). Convolution gives us

\[
y[n] = \sum_{m=-\infty}^{\infty} e^{j\omega(n-m)} h(m)
\]

\[
= \sum_{m=0}^{\infty} e^{j\omega(n-m)} \alpha^m
\]

\[
= e^{j\omega n} \sum_{m=0}^{\infty} (e^{-j\omega} \alpha)^m
\]

\[
= e^{j\omega n} \frac{1}{1 - e^{-j\omega} \alpha}
\]

Thus the amplitude of the output is given by

\[
\left| \frac{1}{1 - e^{-j\omega} \alpha} \right| = \left| \frac{1}{1 - \alpha \cos \omega + j\alpha \sin \omega} \right|
\]

\[
= \left| \frac{1}{1 - \alpha \cos \omega - j\alpha \sin \omega} \right| = \frac{1}{\sqrt{(1 - \alpha \cos \omega)^2 + (\alpha \sin \omega)^2}}
\]

Thus the output is given by

\[
y[n] = \frac{1}{\sqrt{(1 - \alpha \cos \omega)^2 + (\alpha \sin \omega)^2}} e^{j(\omega n + \phi)}
\]

where

\[
\tan \phi = \frac{-\alpha \sin \omega}{1 - \alpha \cos \omega}
\]

(d) The input signal is \( \text{Im}(e^{j\omega n}) \). Thus the output is given by

\[
y[n] = \sum_{m=-\infty}^{\infty} \text{Im}(e^{j\omega(n-m)}) h(m)
\]

\[
= \text{Im}(\sum_{m=-\infty}^{\infty} e^{j\omega(n-m)} h(m))
\]

\[
= \text{Im}(\sum_{m=0}^{\infty} e^{j\omega(n-m)} h(m))
\]
The second equality follows from the linearity of the $\text{Im}(\cdot)$ function. Using the solution to the part above we get

$$y[n] = \text{Im}\left(\frac{1}{\sqrt{(1 - \alpha \cos \omega)^2 + (\alpha \sin \omega)^2}} e^{j(\omega n + \phi)}\right)$$

$$= \frac{1}{\sqrt{(1 - \alpha \cos \omega)^2 + (\alpha \sin \omega)^2}} \sin(\omega n + \phi)$$

where $\phi$ is given in the solution to the previous part.

2. The amplitude is given by

$$\frac{1}{\sqrt{(1 - \alpha \cos \omega)^2 + (\alpha \sin \omega)^2}}$$

For $\omega = 0$ the amplitude is $\frac{1}{1-\alpha}$. For $\omega = \pi$ the amplitude is $\frac{1}{1+\alpha}$.

3. As $\alpha \to 1$ the amplitude at $\omega = 0$ tends to infinity and amplitude at $\omega = \pi$ tends to $\frac{1}{2}$. This is a like a low-pass filter.