Homework Set # 4
Principles of Wireless Networks

Problem 1 (Min Cut-Rank Relation)

Consider a linear deterministic layered network. Let $\mathbf{G}_{ij}$ be the $q \times q$ channel matrix from node i to node j, and

$$\mathbf{y}_j[t] = \sum_i \mathbf{G}_{ij}\mathbf{x}_i[t],$$

where $\mathbf{x}_i[t]$ and $\mathbf{y}_j[t]$ are the transmitted vector from node i and received vector at node j, respectively. The relay nodes wait till they receive T vectors and then they perform their own operations $\mathbf{F}_j$ to obtain the transmitting vectors in the next block.

Note that the end-to-end transfer function $\mathbf{H}$ is formed as a combination of channel matrices $\mathbf{I}_T \otimes \mathbf{G}_{ij}$, and the operations performed at the relays $\mathbf{F}_j$,

$$\mathbf{Y}_D = \mathbf{H}\mathbf{X}_S.$$

Show that for uniform random choices of the relay operations, we have

$$\Pr[\text{rank}(\mathbf{H}) < T(\min_{\Omega} \text{rank}(\mathbf{G}_{\Omega,\Omega^c}) - \varepsilon)] < 2^{|V|}2^{-T\varepsilon},$$

where $|V|$ is the number of nodes in the network.

Problem 2 (Multi-Sources Deterministic Network)

Consider a linear deterministic network with two sources S_1 and S_2, where S_i wants to communicate to the destination node at rate R_1. Assume the network is layered with respect to both sources. Here We will derive an inner bound (achievable region) for $\mathcal{R} = \{(R_1,R_2) : (R_1,R_2) \text{ is achievable}\}$, using the following scheme.

Fix a block length T and an arbitrary product distribution $\prod_{i\in\mathcal{V}} p(x_i)$. The source nodes maps their messages $w_i \in \{1,2,\ldots,2^{TR_i}\}$ to transmitting sequence of vectors $\mathbf{x}_i[t]$, $t = 1,\ldots,T$ chosen uniformly at random according $p(x_{S_i})$ (random codebook). Having T vectors received at the relay node i, it maps its received sequence $\mathbf{y}_i$ to $\mathbf{x}_i$ where $\mathbf{x}_i$ is chosen uniformly at random according to $p(x_i)$ (random mapping operation). In the following we will show that the error probability of this scheme is vanishing as T grows.

(a). Justify the following chain of equality and inequalites.

$$\begin{aligned}
\mathbb{E}[\Pr(\text{err})] &\overset{(i)}{=} \mathbb{E}[\Pr(\text{error at } D|(w_1,w_2) \text{ is sent})] \\
&\overset{(ii)}{\leq} \sum_{(w_1',w_2')\neq(w_1,w_2)} \mathbb{E}[\Pr(D \text{ cannot distinguish } (w_1',w_2') \text{ and } (w_1,w_2)|(w_1,w_2) \text{ is sent})] \\
&= \sum_{w_1'\neq w_1} \mathbb{E}[\Pr(D \text{ cannot distinguish } (w_1',w_2) \text{ and } (w_1,w_2)|(w_1,w_2) \text{ is sent})] \\
&\quad + \sum_{w_2'\neq w_2} \mathbb{E}[\Pr(D \text{ cannot distinguish } (w_1,w_2') \text{ and } (w_1,w_2)|(w_1,w_2) \text{ is sent})] \\
&\quad + \sum_{\substack{w_1'\neq w_1\\ w_2'\neq w_2}} \mathbb{E}[\Pr(D \text{ cannot distinguish } (w_1',w_2') \text{ and } (w_1,w_2)|(w_1,w_2) \text{ were sent})] \quad (1)
\end{aligned}$$

(b). We can continue by bounding each of the terms in the RHS of above inequality. For each $\emptyset \neq I \subseteq \{1,2\}$, we define $\mathcal{S}_I = \{S_i : i \in I\}$ and $W_I = \{w_i : i \in I\}$. Show that

$$\begin{aligned}
&\mathbb{E}[\Pr(D \text{ cannot distinguish } (W_{I^c}, W_I') \text{ and } (W_{I^c}, W_I)|(w_1,w_2) \text{ is sent})] \\
&\overset{(i)}{\leq} \sum_{\Omega \in \Lambda_I} \mathbb{E}\left[\Pr\left(\substack{\Omega \text{ can distinguish } (W_{I^c}, W_I') \text{ and } (W_{I^c}, W_I) \\ \text{but } \Omega^c \text{ cannot}}|(w_1,w_2) \text{ were sent}\right)\right] \\
&\overset{(ii)}{\leq} \sum_{\Omega \in \tilde{\Lambda}_I} \mathbb{E}\left[\Pr\left(\substack{\Omega \text{ can distinguish } (W_{I^c}, W_I') \text{ and } (W_{I^c}, W_I) \\ \text{but } \Omega^c \text{ cannot}}|(w_1,w_2) \text{ is sent}\right)\right] \\
&\overset{(iii)}{\leq} \sum_{\Omega \in \tilde{\Lambda}_I} \mathbb{E}\left[\Pr\left(\Omega^c \text{ cannot distinguish } (W_{I^c}, W_I') \text{ and } (W_{I^c}, W_I)|_{\Omega \text{ can distinguish } (W_{I^c}, W_I') \text{ and } (W_{I^c}, W_I)}^{(w_1,w_2) \text{ is sent}}\right)\right]
\end{aligned}$$

where $\Lambda_I = \{\Omega : \mathcal{S}_I \subseteq \Omega, D \in \Omega^c\}$, $\tilde{\Lambda}_I = \{\Omega : \mathcal{S}_I \subseteq \Omega, \mathcal{N}(\mathcal{S}_I) \cup \{D\} \subseteq \Omega^c\}$, and $\mathcal{N}(\mathcal{S}_I)$ is the subset of the nodes in the network that are not in the flow of $\mathcal{S}_I$, *i.e,* set of all nodes j such that there is no path from any source node in $\mathcal{S}_I$ to j.

(c). Argue that

$$\mathbb{E}\left[\Pr\left(\Omega^c \text{ cannot distinguish } (W_{I^c}, W_I') \text{ and } (W_{I^c}, W_I)|_{\Omega \text{ can distinguish } (W_{I^c}, W_I') \text{ and } (W_{I^c}, W_I)}^{(w_1,w_2) \text{ is sent}}\right)\right] \leq 2^{-TH(Y_{\Omega^c}|X_{\Omega^c})}.$$

(d). Note that the inner term in each of the summations in RHS of (1), does not depend on W_{I^c}'. Count the number of terms appear in each summation.

(e). By summarizing the above inequality, show that

$$\begin{aligned}
\mathbb{E}[\Pr(\text{err})] \leq\ & |\tilde{\Lambda}_{\{S_1\}}| 2^{TR_1} 2^{-T \min_{\Omega \in \tilde{\Lambda}_{\{S_1\}}} H(Y_{\Omega^c}|X_{\Omega^c})} \\
&+ |\tilde{\Lambda}_{\{S_2\}}| 2^{TR_2} 2^{-T \min_{\Omega \in \tilde{\Lambda}_{\{S_2\}}} H(Y_{\Omega^c}|X_{\Omega^c})} \\
&+ |\tilde{\Lambda}_{\{S_1,S_2\}}| 2^{T(R_1+R_2)} 2^{-T \min_{\Omega \in \tilde{\Lambda}_{\{S_1,S_2\}}} H(Y_{\Omega^c}|X_{\Omega^c})}.
\end{aligned}$$

(f). Find constraints on (R_1, R_2) such that $\mathbb{E}[\Pr(\text{err})] \to 0$ as $T \to \infty$.

(g). The constraints you found in part (f) involve $\tilde{\Lambda}_I$. In this part we show that $\tilde{\Lambda}_I$ can be replaced by Λ_I by showing

$$\min_{\Omega \in \tilde{\Lambda}_I} H(Y_{\Omega^c}|X_{\Omega^c}) = \min_{\Omega \in \Lambda_I} H(Y_{\Omega^c}|X_{\Omega^c}) \tag{2}$$

for any distribution $p(x_i)$ and all I. First argue that the RHS of (2) does not exceed the LHS. Now, assume that Ω^* is the minimizer of the RHS and $\Omega^* \cap \mathcal{N}(I) \neq \emptyset$. Define $\Omega' = \Omega^* \setminus \mathcal{N}(I)$. Show that $\Omega' \in \Lambda_I$ and it also minimizes the RHS by justifying the following inequalities.

$$\begin{aligned}
H(Y_{\Omega'^c}|X_{\Omega'^c}) &\overset{(i)}{=} H(Y_{\Omega^{*c}}|X_{\Omega'^c}) + H(Y_{\mathcal{N}(I)\setminus\Omega^{*c}}|Y_{\Omega^{*c}} X_{\Omega'^c}) \\
&\overset{(ii)}{=} H(Y_{\Omega^{*c}}|X_{\Omega'^c}) \\
&\overset{(iii)}{\leq} H(Y_{\Omega^{*c}}|X_{\Omega^{*c}}) \\
&\overset{(iv)}{=} \min_{\Omega \in \Lambda_I} H(Y_{\Omega^c}|X_{\Omega^c}).
\end{aligned}$$

Problem 3 (Typicality in deterministic channels)

We define robust typicality as follows.

We define $\underline{x} \in T_\delta$ if

$$|\nu_{\underline{x}}(x) - p(x)| \leq \delta p(x), \ \forall x \in \mathcal{X}$$

where $\nu_{\underline{x}}(x) = \frac{1}{T}|\{t : x_t = x\}|$, is the empirical frequency.

We define robust joint typicality in the natural way as follows.
We define $(\underline{x}, \underline{y}) \in T_\delta$ if

$$|\nu_{\underline{x},\underline{y}}(x,y) - p(x,y)| \leq \delta p(x,y), \ \forall (x,y) \in \mathcal{X} \times \mathcal{Y}$$

where $\nu_{\underline{x},\underline{y}}(x,y) = \frac{1}{T}|\{t : (x_t, y_t) = (x,y)\}|$, is the joint empirical frequency.
Suppose we have a deterministic channel, $Y = f(X)$.

(a). Show that if $\underline{x} \in T_\delta$ and

$$\underline{y} = [f(x_1), \ldots, f(x_n)]$$

then $(\underline{x}, \underline{y}) \in T_\delta$ and $\underline{y} \in T_\delta$.

(b). For $Y = f(X)$, *i.e.,* if

$$p(y|x=a) = \begin{cases} 1 & \text{if } y = f(a) \\ 0 & \text{else} \end{cases}$$

Show that if $(\underline{x}, \underline{y}) \in T_\delta$ it implies that $\underline{x} \in T_\delta$ and $\underline{y} = [f(x_1), \ldots, f(x_n)]$.