Solutions: Homework Set # 2
Principles of Wireless Networks

Problem 1 (Coherent capacity: Symmetric assumption)

(a). The capacity of the MIMO channel with receiver CSI is given by

$$\begin{aligned}
C &= \max_{\mathbf{K}_x:\mathrm{Tr}(\mathbf{K}_x)\le P}\mathbb{E}\left[\log\det\left(I_{M_r}+\frac{1}{N_0}\mathbf{H}\mathbf{K}_x\mathbf{H}^*\right)\right]\\
&= \max_{\mathbf{K}_x:\mathrm{Tr}(\mathbf{K}_x)\le P}\mathbb{E}\left[\log\det\left(I_{M_r}+\frac{1}{N_0}\mathbf{U}_r\mathbf{H}^{\mathbf{a}}\mathbf{U}_t^*\mathbf{K}_x\mathbf{U}_t\mathbf{H}^{\mathbf{a}*}\mathbf{U}_r^*\right)\right]\\
&\overset{(i)}{=} \max_{\mathbf{K}_x:\mathrm{Tr}(\mathbf{K}_x)\le P}\mathbb{E}\left[\log\det\left(I_{M_r}+\frac{1}{N_0}\mathbf{H}^{\mathbf{a}}\mathbf{U}_t^*\mathbf{K}_x\mathbf{U}_t\mathbf{H}^{\mathbf{a}*}\mathbf{U}_r^*\mathbf{U}_r\right)\right]\\
&= \max_{\mathbf{K}_x:\mathrm{Tr}(\mathbf{K}_x)\le P}\mathbb{E}\left[\log\det\left(I_{M_r}+\frac{1}{N_0}\mathbf{H}^{\mathbf{a}}\mathbf{U}_t^*\mathbf{K}_x\mathbf{U}_t\mathbf{H}^{\mathbf{a}*}\right)\right]
\end{aligned}$$

(i) follows from the identity $\det(I+AB)=\det(I+BA)$.

(b). We can always consider the covariance matrix of the form $\mathbf{K}_x=\mathbf{U}_t\tilde{\mathbf{K}}_x\mathbf{U}_t^*$ where $\tilde{\mathbf{K}}_x$ is also a covariance matrix satisfying the total power constraint.

$$\begin{aligned}
C &= \max_{\tilde{\mathbf{K}}_x:\mathrm{Tr}(\tilde{\mathbf{K}}_x)\le P}\mathbb{E}\left[\log\det\left(I_{M_r}+\frac{1}{N_0}\mathbf{H}^{\mathbf{a}}\mathbf{U}_t^*\mathbf{K}_x\mathbf{U}_t\mathbf{H}^{\mathbf{a}*}\right)\right]\\
&= \max_{\tilde{\mathbf{K}}_x:\mathrm{Tr}(\tilde{\mathbf{K}}_x)\le P}\mathbb{E}\left[\log\det\left(I_{M_r}+\frac{1}{N_0}\mathbf{H}^{\mathbf{a}}\tilde{\mathbf{K}}_x\mathbf{H}^{\mathbf{a}*}\right)\right]
\end{aligned}$$

Define a diagonal matrix Π_i with -1 in the i^{th} position and 1 in the remaining positions. The entries of $\Pi_i\tilde{\mathbf{K}}_x\Pi_i^*$ equal those of $\tilde{\mathbf{K}}_x$ except in the off diagonal positions in the i^{th} row and the i^{th} column where the sign is reversed. The matrix $\Pi_i\tilde{\mathbf{K}}_x\Pi_i^*$ is a covariance matrix satisying the power constraint, i.e., $\mathrm{Tr}\{\Pi_i\tilde{\mathbf{K}}_x\Pi_i^*\}=\mathrm{Tr}\{\tilde{\mathbf{K}}_x\}$. If we denote $R(\tilde{\mathbf{K}}_x)$ to be

$$R(\tilde{\mathbf{K}}_x)=\mathbb{E}\left[\log\det\left(I_{M_r}+\frac{1}{N_0}\mathbf{H}^{\mathbf{a}}\tilde{\mathbf{K}}_x\mathbf{H}^{\mathbf{a}*}\right)\right],$$

then

$$\begin{aligned}
R(\Pi_i\tilde{\mathbf{K}}_x\Pi_i^*) &= \mathbb{E}\left[\log\det\left(I_{M_r}+\frac{1}{N_0}\mathbf{H}^{\mathbf{a}}\Pi_i\tilde{\mathbf{K}}_x\Pi_i^*\mathbf{H}^{\mathbf{a}*}\right)\right]\\
&= \mathbb{E}\left[\log\det\left(I_{M_r}+\frac{1}{N_0}(\mathbf{H}^{\mathbf{a}}\Pi_i)\tilde{\mathbf{K}}_x(\mathbf{H}^{\mathbf{a}}\Pi_i)^*\right)\right]\\
&\overset{(ii)}{=} \mathbb{E}\left[\log\det\left(I_{M_r}+\frac{1}{N_0}\mathbf{H}^{\mathbf{a}}\tilde{\mathbf{K}}_x\mathbf{H}^{\mathbf{a}*}\right)\right]\\
&= R(\tilde{\mathbf{K}}_x)
\end{aligned}$$

where (ii) follows from the fact that, since the columns of $\mathbf{H}^{\mathbf{a}}$ are independent and their distribution symmetric, $\mathbf{H}^{\mathbf{a}}$ and $\mathbf{H}^{\mathbf{a}}\Pi_i$ have the same distribution. From the concavity of the $\log\det(\cdot)$

function, it follows that

$$
\begin{aligned}
R(\tilde{\mathbf{K}}_x) &= \frac{1}{2}R(\Pi_i\tilde{\mathbf{K}}_x\Pi_i^*) + \frac{1}{2}R(\tilde{\mathbf{K}}_x) \\
&= R(\frac{1}{2}(\tilde{\mathbf{K}}_x + \Pi_i\tilde{\mathbf{K}}_x\Pi_i^*))
\end{aligned}
$$

The entries of the matrix $\frac{1}{2}(\tilde{\mathbf{K}}_x + \Pi_i\tilde{\mathbf{K}}_x\Pi_i^*)$ are equal to those in $\tilde{\mathbf{K}}_x$ except in the off diagonal positions in the i^{th} row and column, where the entries are zero. Iterating the above process M_t times for $i = 1, \ldots, M_t$, we find that the optimal $\tilde{\mathbf{K}}_x$ is diagonal which proves our claim.

Problem 2 (Universal code design criterion for the MISO channel)

(a). The $Q(\cdot)$ function is decreasing in its argument. The error probability is maximum for the $\mathbf{h}$ for which $\|\mathbf{h}^*(\mathbf{X}_A - \mathbf{X}_B)\|$ is minimum subject to $\|\mathbf{h}\|^2 \geq \frac{M_t(2^R-1)}{SNR}$

$$
\begin{aligned}
\|\mathbf{h}^*(\mathbf{X}_A - \mathbf{X}_B)\|^2 &= \mathbf{h}^*(\mathbf{X}_A - \mathbf{X}_B)(\mathbf{X}_A - \mathbf{X}_B)^*\mathbf{h} \\
&= \mathbf{h}^*\mathbf{U}\Lambda_{A-B}\mathbf{U}^*\mathbf{h} \\
&= \tilde{\mathbf{h}}^*\Lambda_{A-B}\tilde{\mathbf{h}}
\end{aligned}
$$

Since $\tilde{\mathbf{h}} = \mathbf{h}^*\mathbf{U}$ is distributed as $\mathbf{h}$,

$$
\begin{aligned}
\min_{\mathbf{h}:\|\mathbf{h}\|^2 \geq \frac{M_t(2^R-1)}{SNR}} \|\mathbf{h}^*(\mathbf{X}_A - \mathbf{X}_B)\|^2 &= \min_{\mathbf{h}:\|\mathbf{h}\|^2 \geq \frac{M_t(2^R-1)}{SNR}} \mathbf{h}^*\Lambda_{A-B}\mathbf{h} \\
&= \frac{M_t(2^R-1)}{SNR} \min_{\mathbf{h}:\|\mathbf{h}\|^2=1} \sum_i |h_i|^2\lambda_i^2 \\
&\geq \frac{M_t(2^R-1)}{SNR}\lambda_1^2
\end{aligned}
$$

where λ_1 is the smallest singular value of $(\mathbf{X}_A - \mathbf{X}_B)$. The minimum error probability is given by

$$
\begin{aligned}
\max_{\mathbf{h}:\|\mathbf{h}\|^2 \geq \frac{M_t(2^R-1)}{SNR}} Q\big(\frac{\|\mathbf{h}^*(\mathbf{X}_A - \mathbf{X}_B)\|}{\sqrt{2}}\big) &= Q\big(\sqrt{\frac{\lambda_1^2 M_t(2^R-1)}{2SNR}}\big) \\
&= Q\big(\sqrt{\frac{1}{2}\tilde{\lambda}_1^2 M_t(2^R-1)}\big)
\end{aligned}
$$

where $\tilde{\lambda}_1$ is the smallest singular value of $\frac{1}{\sqrt{SNR}}(\mathbf{X}_A - \mathbf{X}_B)$.

(b).

$$
\begin{aligned}
Q\big(\sqrt{\frac{\lambda_1^2 M_t(2^R-1)}{2SNR}}\big) &< e^{-\frac{\lambda_1^2 M_t(2^R-1)}{4SNR}} \\
&\approx e^{-\lambda_1^2 SNR^{-(1-r)}}
\end{aligned}
$$

where the approximation is made on the scale of SNR. As long as $\lambda_1^2 > SNR^{1-r}$, the error probability goes down exponentially with SNR.

Problem 3 (Diversity-Multiplexing tradeoff - Alamouti scheme over the $2 \times M_r$ MIMO)

(a). The received vector at the first time instant is given by

$$\mathbf{y}[1] = \mathbf{h}_1 u_1 + \mathbf{h}_2 u_2 + \mathbf{z}[1]$$

and at the second time instant is given by

$$\mathbf{y}[2] = \mathbf{h}_1(-u_2^*) + \mathbf{h}_2 u_1^* + \mathbf{z}[2]$$

This can be rewritten as

$$\begin{pmatrix} \mathbf{y}[1] \\ (\mathbf{y}[2]^*)^\top \end{pmatrix} = \begin{pmatrix} \mathbf{h}_1 & \mathbf{h}_2 \\ (\mathbf{h}_2^*)^\top & -(\mathbf{h}_1^*)^\top \end{pmatrix} \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} + \begin{pmatrix} \mathbf{z}[1] \\ (\mathbf{z}[2]^*)^\top \end{pmatrix}$$

(b). Define $\mathbf{H}$ to be the matrix with columns $\mathbf{h}_1$ and $\mathbf{h}_2$ and let $\|\mathbf{H}\|^2 = \|\mathbf{h}_1\|^2 + \|\mathbf{h}_2\|^2$ Projecting the output along the direction of $\begin{pmatrix} \mathbf{h}_1 \\ (\mathbf{h}_2^*)^\top \end{pmatrix}$ gives

$$\begin{aligned} r_1 &= \frac{1}{\|\mathbf{H}\|} \begin{pmatrix} \mathbf{h}_1^* & (\mathbf{h}_2)^\top \end{pmatrix} \begin{pmatrix} \mathbf{y}[1] \\ (\mathbf{y}[2]^*)^\top \end{pmatrix} \\ &= \|\mathbf{H}\| u_1 + w_1 \end{aligned}$$

where $w_1 \sim \mathcal{C}\eta(0,1)$. Likewise projecting the output along the direction of $\begin{pmatrix} \mathbf{h}_2 \\ -(\mathbf{h}_1^*)^\top \end{pmatrix}$ gives

$$\begin{aligned} r_2 &= \frac{1}{\|\mathbf{H}\|} \begin{pmatrix} \mathbf{h}_2^* & -(\mathbf{h}_1)^\top \end{pmatrix} \begin{pmatrix} \mathbf{y}[1] \\ (\mathbf{y}[2]^*)^\top \end{pmatrix} \\ &= \|\mathbf{H}\| u_2 + w_2 \end{aligned}$$

where $w_2 \sim \mathcal{C}\eta(0,1)$. We have made use of the fact that the two columns of $\mathbf{H}$ are orthogonal to separate the signals u_1 and u_2 at the receiver.

(c). The channel corresponding to either stream u_i is a scalar channel with gain $\|\mathbf{H}\|$ and by reasoning similar to the previous two questions, the diversity gain at rate $r \log SNR$ is given by $2M_r(1-r)$.

Problem 4 (Diversity-Multiplexing tradeoff - Repetition coding over L parallel channels)

The output of the i^{th} channel is given by

$$y_i = h_i u + z_i$$

Collecting the L outputs, we have

$$\mathbf{y} = \begin{pmatrix} y_1 \\ y_2 \\ .. \\ .. \\ y_L \end{pmatrix} = \begin{pmatrix} h_1 \\ h_2 \\ .. \\ .. \\ h_L \end{pmatrix} u + \begin{pmatrix} z_1 \\ z_2 \\ .. \\ .. \\ z_L \end{pmatrix}$$

Projecting the output vector in the direction of $\begin{pmatrix} h_1 \\ h_2 \\ .. \\ .. \\ h_L \end{pmatrix}$ gives

$$\begin{aligned} \tilde{y} &= \frac{1}{\|\mathbf{h}\|} \begin{pmatrix} h_1^* & h_2^* & .. & .. & h_L^* \end{pmatrix} \mathbf{y} \\ &= \|\mathbf{h}\| u + \tilde{z} \end{aligned}$$

where $\tilde{z} \sim \mathcal{C}\eta(0,1)$ and $\|\mathbf{h}\|^2 = \sum_l |h_l|^2$. The outage probability at rate $r \log SNR$ for this effective scalar channel is given by

$$\begin{aligned} \Pr\left\{ \log(1 + \|\mathbf{h}\|^2 SNR) < r \log SNR \right\} &= \Pr\left\{ \|\mathbf{h}\|^2 < \frac{SNR^r - 1}{SNR} \right\} \\ &\approx \Pr\left\{ \|\mathbf{h}\|^2 < SNR^{-(1-r)} \right\} \\ &\approx SNR^{-L(1-r)} \end{aligned}$$

where the two approximations follow for large enough SNR and since $\|\mathbf{h}\|^2 \sim \chi^2_{2L}$, so $\Pr(\|\mathbf{h}\|^2 < \varepsilon) \approx \varepsilon^L$. Since r is the rate achievable over L channel uses, the effective rate $\tilde{r} = \frac{r}{L}$. In terms of this effective rate, the diversity gain is given by $L(1 - L\tilde{r})$.

Problem 5 (Diversity-Multiplexing tradeoff - V-Blast with annuling)

The output at the receiver is given by

$$\mathbf{y} = \mathbf{h}_k x_k + \sum_{i \neq k} \mathbf{h}_i x_i + \mathbf{z}$$

The decorrelator projects the output in the subspace orthogonal to the columns $\{\mathbf{h}_i\}_{i \neq k}$. If we call the projection matrix $\mathbf{Q}_k$, the projection is given by

$$\begin{aligned} \tilde{\mathbf{y}}_k &= \mathbf{Q}_k \mathbf{y} \\ &= \mathbf{Q}_k \mathbf{h}_k x_k + \mathbf{Q}_k \mathbf{z} \end{aligned}$$

Projecting $\tilde{\mathbf{y}}_k$ along $\mathbf{Q}_k \mathbf{h}_k$ gives the equivalent scalar channel where the achievable rate per stream k is given by $\log(1 + \frac{SNR}{n_t} \|\mathbf{Q}_k \mathbf{h}_k\|^2)$. In problem 3 of homework 2, we saw that $\mathbf{Q}_k$ has rank $n_r - (n_t - 1)$. Therefore $\|\mathbf{Q}_k \mathbf{h}_k\|^2 \sim \chi^2_{2(n_r - n_t + 1)}$. Therefore the diversity gain at multiplexing gain of r_k is given by $(n_r - n_t + 1)(1 - r_k)$. Since we assume the streams to have equal rate, the net rate $r = \sum_k r_k$, or equivalently, $r_k = \frac{r}{n_t}$. So the diversity gain is equivalently given by $(n_r - n_t + 1)(1 - \frac{r}{n_t})$.

Problem 6 (Diversity multiplexing tradeoff using superposition codes)

(a). We can assume that $T \to \infty$, and therefore get the D-M tradeoff $d(r) = 1 - r$. Note that in fact we do not need T to be too large. As we have seen in the class uncoded QAM achieves the D-M tradeoff of this channel with $T = 1$.

(b).

$$\begin{aligned} P_{\text{out}}(r_H, r_L, \text{SNR}) &= \Pr\left[\log\left(1 + \text{SNR}^{1-\beta} |h^{(b)}|^2 + \text{SNR} |h^{(b)}|^2 \right) < r_L \log \text{SNR} + r_H \log \text{SNR} \right] \\ &= \Pr\left[|h^{(b)}|^2 < \frac{\text{SNR}^{r_L + r_H} - 1}{\text{SNR} + \text{SNR}^{1-\beta}} \right] \\ &\doteq \text{SNR}^{-(1 - r_L - r_H)}. \end{aligned}$$

Therefore, $\tilde{d}(r_L, r_H) = 1 - r_L - r_H$.

(c). Since we use successive decoder, we have to consider the weak message as noise when we decode the first one. Let $|h^{(b)}|^2 \doteq \text{SNR}^{-\alpha}$ for some $\alpha \in \mathbb{R}$. Therefore we have

$$\begin{aligned}
\text{SINR}_H &= \frac{\text{SNR}|h^{(b)}|^2}{\text{SNR}^{1-\beta}|h^{(b)}|^2+1} \\
&= \frac{\text{SNR}^{1-\alpha}}{\text{SNR}^{1-\beta-\alpha}+1} \\
&\doteq \begin{cases} \text{SNR}^{\beta} & \text{if } 1-\alpha-\beta > 0 \\ \text{SNR}^{1-\alpha} & \text{if } 1-\alpha-\beta \leq 0 \end{cases} \\
&= \text{SNR}^{\min(1-\alpha,\beta)}.
\end{aligned}$$

Hence,

$$\begin{aligned}
P_{\text{out}}(r_H, \text{SNR}) &= \Pr\left[\log\left(1+\text{SINR}_H\right) < r_H \log \text{SNR}\right] \\
&= \Pr\left[\log\left(1+\text{SNR}^{\min(1-\alpha,\beta)}\right) < r_H \log \text{SNR}\right] \\
&= \Pr\left[\log\left(1+\text{SNR}^{\beta}\right) < r_H \log \text{SNR} \middle| \alpha < 1-\beta\right] \cdot \Pr[\alpha < 1-\beta] \\
&\quad + \Pr\left[\log\left(1+\text{SNR}^{1-\alpha}\right) < r_H \log \text{SNR}, \alpha > 1-\beta\right]
\end{aligned}$$

It is clear that for $\beta = 1$, we get

$$P_{\text{out}}(r_H, \text{SNR}) = \Pr\left[\log\left(1+\text{SNR}^{1-\alpha}\right) < r_H \log \text{SNR}\right] \doteq \text{SNR}^{-(1-r_H)}.$$

For $\beta < 1$, we can write

$$\begin{aligned}
P_{\text{out}}(r_H, \text{SNR}) &= \mathbf{1}_{[\beta < r_H]}\left[1-\text{SNR}^{-(1-\beta)}\right] + \text{SNR}^{-\max(1-r_H, 1-\beta)} \\
&\doteq \begin{cases} 1 & \text{if } r_H > \beta \\ \text{SNR}^{-(1-r_H)} & \text{if } r_H \leq \beta. \end{cases}
\end{aligned}$$

(d). It is clear that

$$d_H = \lim_{SNR \to \infty} \frac{\log P_{\text{out}}(M_H, SNR)}{\log SNR} = \begin{cases} 0 & \text{if } r_H > \beta \\ 1-r_H & \text{if } r_H > \beta. \end{cases}$$

(e). For $\beta > r_H$, we have $d_H = 1 - r_H$, which is the same as in part (a).