

Homework Set #7

Due Thursday, 27 November 2008, before 12:00 noon, INR 031/032/038

Problem 1 (JOINTLY TYPICAL SEQUENCES)

We consider a binary symmetric channel (BSC) with crossover probability $p = 0.1$.

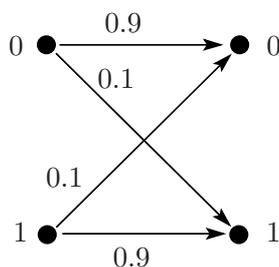


Figure 1: Binary symmetric channel with crossover probability 0.1

The input distribution that achieves capacity is the uniform distribution (i.e. $p(x) = (\frac{1}{2}, \frac{1}{2})$), which yields the joint distribution $p(x, y)$

x, y	0,0	0,1	1,0	1,1
$p(x, y)$	0.45	0.05	0.05	0.45

The marginal distribution of Y is also $(\frac{1}{2}, \frac{1}{2})$.

- (a) The jointly typical set $A_\epsilon^{(n)}(X, Y)$ is defined as the set of sequences $x^n \in \{0, 1\}^n, y^n \in \{0, 1\}^n$ that satisfy equations

$$\left| -\frac{1}{n} \log p(x^n) - H(X) \right| < \epsilon \tag{1}$$

$$\left| -\frac{1}{n} \log p(y^n) - H(Y) \right| < \epsilon \tag{2}$$

$$\left| -\frac{1}{n} \log p(x^n, y^n) - H(X, Y) \right| < \epsilon. \tag{3}$$

The first two equations correspond to the conditions that x^n and y^n are in $A_\epsilon^{(n)}(X)$ and $A_\epsilon^{(n)}(Y)$, respectively. Consider the last condition, which can be rewritten as

$$-\frac{1}{n} \log p(x^n, y^n) \in (H(X, Y) - \epsilon, H(X, Y) + \epsilon).$$

Let k be the number of different places in which the sequence x^n differs from y^n (k is a function of the two sequences). Then we can write

$$\begin{aligned} p(x^n, y^n) &= \prod_{i=1}^n p(x_i, y_i) \\ &= (0.45)^{n-k} (0.05)^k \\ &= \left(\frac{1}{2}\right)^n (1-p)^{n-k} p^k. \end{aligned}$$

An alternative way of looking at this probability is to look at the binary symmetric channel as an additive channel $Y = X \oplus Z$, where Z is a binary random variable that is equal to 1 with probability p , and is independent of X . In that case,

$$\begin{aligned} p(x^n, y^n) &= p(x^n) p(y^n | x^n) \\ &= p(x^n) p(z^n | x^n) \\ &= p(x^n) p(z^n) \\ &= \left(\frac{1}{2}\right)^n (1-p)^{n-k} p^k. \end{aligned}$$

Show that the condition that (x^n, y^n) are jointly typical is equivalent to the condition that x^n is typical and $z^n = y^n - x^n$ is typical.

- (b) Now consider random coding for the additive channel. As in the proof of the channel coding theorem, assume that 2^{nR} codewords $X^n(1), X^n(2), \dots, X^n(2^{nR})$ are chosen uniformly over the 2^n possible binary sequences of length n . One of these codewords is chosen and sent over the channel. The receiver looks at the received sequence and tries to find a codeword in the code that is jointly typical with the received sequence. This corresponds to finding a codeword $X^n(i)$ such that $Y^n - X^n(i) \in A_\epsilon^{(n)}(Z)$. For a fixed codeword $x^n(i)$, what is the probability that the received sequence Y^n is such that $(x^n(i), Y^n)$ is jointly typical?

- (c) Suppose now that a fixed codeword $x^n(1)$ was sent and Y^n was received. Denote the event E_j :

$$E_j = \{(X^n(j), Y^n) \in A_\epsilon^{(n)}(X, Y)\}, \quad j \in \{2, 3, \dots, 2^{nR}\}$$

to be the event that there is a codeword other than $x^n(1)$ jointly typical with Y^n . Use the union bound to find an upper bound to $Pr\{E_2 \cup E_3 \cup \dots \cup E_{2^{nR}} | x^n(1) \text{ was sent}\}$.

- (d) Given that a particular codeword was sent, the probability of error (averaged over the probability distribution of the channel and over the random choice of codewords) can be rewritten as

$$Pr(\text{error} | x^n(1) \text{ sent}) = \sum_{y^n: y^n \text{ causes error}} p(y^n | x^n(1)).$$

There are two kinds of error: the first occurs if the received sequence y^n is not jointly typical with the transmitted codeword, and the second occurs if there is another codeword jointly typical with the received sequence. Using the results of the preceding parts, calculate this probability of error. By the symmetry of the random coding argument, this does not depend on which codeword was sent.

Problem 2 (THE GILBERT-ELLIOTT CHANNEL)

In this problem we study a time varying problem. This channel has two states, B (bad) and G (good), and binary input alphabet $\mathcal{X} = \{0, 1\}$ and quaternary output alphabet $\mathcal{Y} = \{0, 1, 2, 3\}$. At each time instance, the channel might be in bad or good state, $S_i \in \{B, G\}$, and acts like a binary symmetric channel as shown in Fig. 2.

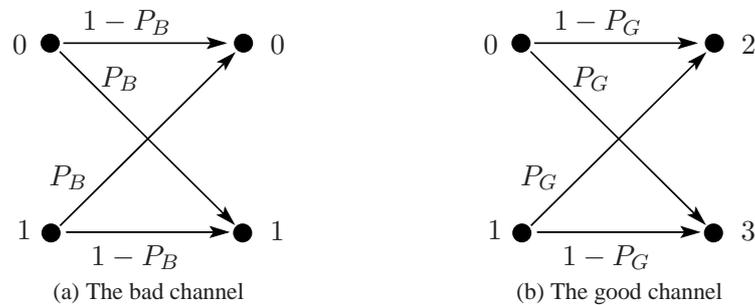


Figure 2: Two binary symmetric channels.

The state of the channel is random process distributed according to the first order Markov process with transition probabilities shown in Fig. 2.

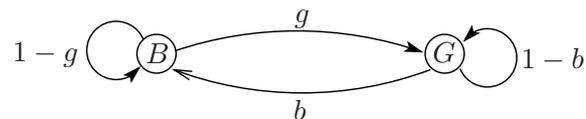


Figure 3: Transition of the states of the channel.

- (a) Find the capacity of each of the channels shown in Fig. 2. What is the optimal input distribution for each of them?
- (b) Compute the stationary distribution of the state of the channel.
- (c) Assume that there is a genie who tells the state of the channel at each time instance to both of the encoder and decoder. Compute the capacity of the channel, C_{SI} (capacity with side information).
- (d) Let the state of the channel be unknown to both encoder and decoder. What can the decoder say about the channel state?
- (e) Compare the capacity of the channel when the states are known to the encoder/decoder, C_{NSI} (capacity without side information) to the capacity obtained in part (c).
Hint: Verify the following inequalities and equalities, and argue whether the inequalities are strict

or tight.

$$\begin{aligned}
C_{NSI} &= \max_{p(x)} \frac{1}{n} I(X^n; Y^n, S^n) \\
&= \max_{p(x)} \frac{1}{n} I(X^n; Y^n | S^n) \\
&= \max_{p(x)} \frac{1}{n} (H(Y^n | S^n) - H(Y^n | X^n, S^n)) \\
&\leq \max_{p(x)} \frac{1}{n} \left(\sum_{i=1}^n H(Y_i | S^n) - \sum_{i=1}^n H(Y_i | X_i, S_i) \right) \\
&\leq \max_{p(x)} \frac{1}{n} \left(\sum_{i=1}^n H(Y_i | S_i) - \sum_{i=1}^n H(Y_i | X_i, S_i) \right) \\
&\leq \max_{p(x)} \frac{1}{n} \sum_{i=1}^n I(X_i; Y_i | S_i)
\end{aligned}$$

Problem 3 (PARALLEL CHANNELS AND CHANNELS WITH TWO INDEPENDENT LOOKS AT Y)

- (a) Consider two discrete memoryless channels $(\mathcal{X}_1, p(y_1|x_1), \mathcal{Y}_1)$ and $(\mathcal{X}_2, p(y_2|x_2), \mathcal{Y}_2)$ with capacities C_1 and C_2 respectively. A new channel $(\mathcal{X}_1 \times \mathcal{X}_2, p(y_1|x_1) \times p(y_2|x_2), \mathcal{Y}_1 \times \mathcal{Y}_2)$ is formed in which $x_1 \in \mathcal{X}_1$ and $x_2 \in \mathcal{X}_2$ are sent simultaneously, resulting in y_1, y_2 . Find the capacity of this channel.
- (b) Let Y_1 and Y_2 be conditionally independent and conditionally identically distributed given X .
1. Show that $I(X; Y_1, Y_2) = 2I(X; Y_1) - I(Y_1; Y_2)$.
 2. Show that the capacity of the channel $(\mathcal{X}, p(y_1, y_2|x), \mathcal{Y}_1 \times \mathcal{Y}_2)$ is less than twice the capacity of the channel $(\mathcal{X}, p(y_1|x), \mathcal{Y}_1)$

Problem 4 (ZERO-ERROR CAPACITY)

A channel with alphabet $\{0, 1, 2, 3, 4\}$ has transition probabilities of the form

$$p(y|x) = \begin{cases} 1/2 & \text{if } y = x \pm 1 \pmod{5} \\ 0 & \text{otherwise} \end{cases} \quad (4)$$

This channel is shown in fig (4).

- (a) Compute the capacity of this channel in bits.
- (b) The zero-error capacity of a channel is the number of bits per channel use that can be transmitted with zero probability of error. Clearly, the zero-error capacity of this channel is at least 1 bit (consider the codebook $\mathcal{C} = \{0, 1\}$ for example). Find a block code that shows that the zero-error capacity is greater than 1 bit. Can you estimate the exact value of the zero-error capacity?
Hint: Consider codes of length 2 for this channel.

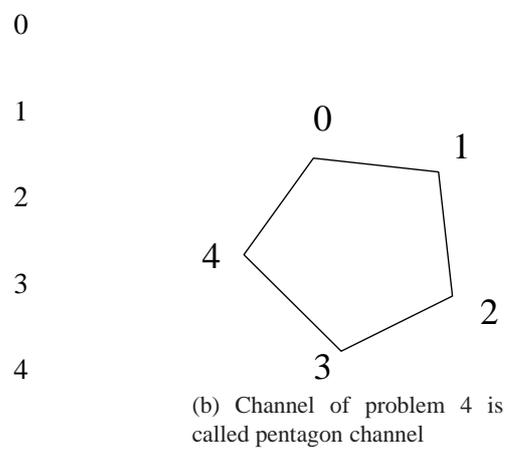
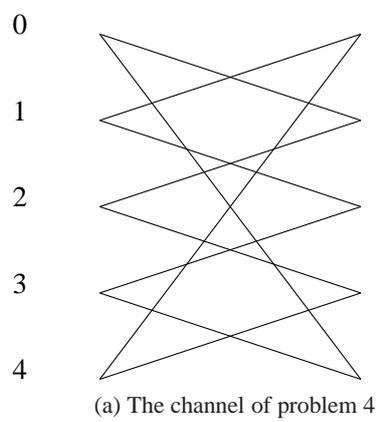


Figure 4: Channel of problem 4.